Quantum Groups and Deformation Quantization: Explicit Approaches and Implicit Aspects

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Abstract

Deformation quantization, which gives a development of quantum mechanics independent of the operator algebra formulation, and quantum groups, which arose from the inverse scattering method and a study of Yang-Baxter equations, share a common idea abstracted earlier in algebraic deformation theory: that algebraic objects have infinitesimal deformations which may point in the direction of certain continuous global deformations, i.e., "quantizations". In deformation quantization the algebraic object is the algebra of "observables" (functions) on symplectic phase space, whose infinitesimal deformation is the Poisson bracket and global deformation a "star product"; in quantum groups it is a Hopf algebra, generally either of functions on a Lie group or (often its dual in the topological vector space sense, as we briefly explain) a completed universal enveloping algebra of a Lie algebra with, for infinitesimal, a matrix satisfying the modified classical Yang-Baxter equation (MCYBE). Frequently existence proofs are known but explicit formulas useful for physical applications have been difficult to extract. One success here comes from "universal deformation formulas" (UDFs), expressions built from a Lie algebra which deform any algebra on which the Lie algebra operates as derivations. The most famous of these is the Moyal product, a special case of a class in which the Lie algebra is abelian. Another comes from recognition that the Belavin-Drinfel'd solutions to the MCYBE are, in fact, infinitesimal deformations for which, in the case of the special linear groups, it is possible to give explicit formulas for the corresponding quantum Yang-Baxter equations. This review paper discusses, necessarily in brief, these and related topics, including "twisting" as a form of UDF and finding formulas for "preferred deformations" of Hopf algebras in which the multiplication or comultiplication is rigid and must be preserved in the course of deformation.

I INTRODUCTION

Mathematics arose as an abstraction of the physical world and thenceforth often became something entirely different. Still, the physical origin of many mathematical notions can be traced and is implicit in many developments. That is what we call *physical mathematics*. On the other hand mathematics is the main language of theoretical physics, albeit used with a very peculiar accent, and the "mathematics toolbox" is crucial in *mathematical physics*. That mutual interaction appears as a watermark throughout many works, including the present paper.

It can be said that quantum groups arose "experimentally" in the Leningrad LOMI group of Ludwig Faddeev around 1980 (see e.g. [KR81, FRT]), during attempts to quantize two dimensional integrable models by methods coming from inverse scattering and a study of what they called Yang-Baxter equations. A posteriori it was discovered that some aspects (e.g. q-special functions in the 19$^{\text{th}}$ century) were present much earlier and that the notion is of importance in various areas of physics, including solid state.

Significant steps forward were made shortly afterwards, in particular when Jimbo [Ji85] systematized these attempts in his study of quantized enveloping algebras. Even more so when Drinfel′d (see e.g. [Dr87]) made explicit the underlying Hopf algebra structures and, relating the dual aspect to deformation quantization, coined the expression "quantum groups."

Deformation quantization was certainly "in the back of the mind" of many almost since the beginning of quantum mechanics ("wave mechanics" in [dB23]) and its avatars developed since 1925 by Heisenberg ("matrix mechanics"), Schrödinger (with his celebrated equation) and especially Weyl with his quantization procedure [We31, Wi32]. But the relation with deformation theory [Ge64] and its role in physics (see e.g. [Fl82]) was made only in [BFFLS].

Already in the formulation of Drinfel′d [Dr87], it is clear that quantum groups are an avatar of deformation quantization, when the category of Hopf algebras is taken into account. But one aspect remained imprecise: the duality between the two aspects, deformations of (functions over) Poisson-Lie groups and quantized enveloping algebras. That was, at least in the compact and semi-simple cases, made clear later [BFGP, BP96], using natural topologies on the corresponding Hopf algebras.

There is another duality which remains largely unexplored. The Gelfand isomorphism theorem, by which commutative algebras can be considered as algebras of functions over some space (their spectrum), expresses a "duality" between a commutative algebra and a topological space. Now, on one hand, commutative algebras are deformed into noncommutative (associative or Hopf) algebras. On the other hand differentiable manifolds, characterized e.g. by some algebraic properties, are deformed into noncommutative ones. And manifolds have symmetries that often can be quantized. These related aspects are so far developed separately, to a large extent. A natural question is thus to study their relations. A very abstract attempt can be found in [KoR]. In view of the present rapid developments in noncommutative manifolds (see e.g. [CDV]) one should probably start with specific examples and see if a pattern arises, hence the need for explicit approaches.

In mathematics an abstract existence proof (such as the one in [EK96]) is perfectly satisfactory, even more so when an algorithmic construction is given (such as in [Fe94]). But for physical applications one needs to perform explicit calculations and few realizations are truly explicit. In the compact case, the ideas underlying [BFGP] were tested in the basic example of $SU(2)$ where explicit formulas were obtained [BFP]. We shall present here that example with this fact in mind. We also present the only known explicit formulas expressing the preferred deformation of a standard quantum algebra. Namely, the preferred $*$-products for the quantum linear spaces associated with the standard q-deformation of the special linear group $SL(n)$. These approaches show that deformation quantization is implicit in most aspects of quantum groups theory.

Now, for deformation quantization, the paradigm of the Moyal star product is explicit enough, and so are a number of related integral formulas in $\mathbb{R}^{2n}$ and a few manifolds. But for general symplectic, even more so Poisson, manifolds one has mainly existence theorems which are not easily made explicit. At its origin 25 years ago and again much more recently [Fr79], some explicit formulas were and are being developed. A somewhat explicit formula for many coadjoint orbits was recently developed [AL03] in terms of a pairing for generalized Verma modules. Part of this paper will be devoted to explicit approaches in that underlying aspect of quantum groups theory. In particular we shall see that quantum groups techniques may produce explicit (strict) deformation quantizations on the basis of what we call universal deformation formulas (UDFs). These come from mathematical entities living within a given algebraic structure S and they produce explicit deformations of any algebra which is also a S-module.

We start this paper (Section II) by a short presentation of deformation theory, mostly following Gerstenhaber. That somewhat arid presentation will be balanced by complements and, in Section III, by a survey of how quantum mechanics *is* a deformation of classical mechanics, with some explicit examples. Section IV begins with the explicit example (the case of $SU_q(2)$) that triggered the general theory (presented afterwards) of both approaches to quantum groups (functions on a Poisson-Lie group and quantized enveloping algebras) as dual topological Hopf algebras, at least in the semi-simple case. The latter theory being (in the general case) not explicit enough we devote Section V to a variety of explicit formulas for deformation quantization (star products), in particular for symmetric spaces and related universal deformation formulas. Section VI discusses the implications of the preceding study for obtaining explicit formulas for quantum groups, via star products or R matrices.

This paper contains only a brief review of some of the rapid developments in deformation quantization since its introduction in [BFFLS]. André Weil's prediction near the end of the last century that deformation theory would be a major topic in the 21$^{\text{st}}$ so far seems justified.

II PRELIMINARIES

II.1 The Gestenhaber theory of deformations of algebras.

A concise formulation of a Gerstenhaber deformation of an algebra (associative, Lie, bialgebra, etc.) is [Ge64, BFGP]:

DEFINITION. A deformation of an algebra A over a field $\mathbb{K}$ with deformation parameter ν is a $\mathbb{K}[[\nu]]$-algebra $\tilde{A}$ such that $\tilde{A}/\nu\tilde{A} \approx A$, where A is here considered as an algebra over $\mathbb{K}[[\nu]]$ by base field extension. Two deformations $\tilde{A}$ and $\tilde{A}'$ are called equivalent if they are isomorphic over $\mathbb{K}[[\nu]]$ (by a deformation which reduces to the identity modulo ν, which will always be tacitly understood). A deformation $\tilde{A}$ is said to be trivial if it is isomorphic to the original algebra A (considered by base field extension as a $\mathbb{K}[[\nu]]$-algebra).

Whenever we consider a topology on A, $\tilde{A}$ is supposed to be topologically free. The above definition can (cf. e.g. [Ko99, KoS]) be extended to operads, so as to apply to the Assoc, Lie, Bialg and maybe Gerst operads, and also to the Hopf category (which can not be described by an operad), all possibly with topologies. In the present mathematical physics paper we shall not probe these sophistications, but the reader should keep such powerful possibilities in mind.

For associative (resp. Lie) algebras, the above definition tells us that there exists a new product $*$ (resp. bracket $[\cdot,\cdot]$) such that the new (deformed) algebra is again associative (resp. Lie). Denoting the original composition laws by ordinary product (resp. $\{\cdot,\cdot\}$) this means that, for $u_1, u_2 \in A$ (we can extend this to $A[[\nu]]$ by $\mathbb{K}[[\nu]]$-linearity) we have:

$$u_1 * u_2 = u_1 u_2 + \sum_{r=1}^{\infty} \nu^r C_r(u_1,u_2) \tag{1}$$

$$[u_1,u_2] = \{u_1,u_2\} + \sum_{r=1}^{\infty} \nu^r B_r(u_1,u_2) \tag{2}$$

where the C_r are Hochschild 2-cochains and the B_r (skew-symmetric) Chevalley-Eilenberg 2-cochains, such that for $u_1,u_2,u_3 \in A$ we have $(u_1*u_2)*u_3 = u_1*(u_2*u_3)$ and $\mathscr{S}[[u_1,u_2],u_3] = 0$, where $\mathscr{S}$ denotes summation over cyclic permutations.

For a (topological) *bialgebra* (an associative algebra A where we have in addition a coproduct $\Delta : A \longrightarrow A \otimes A$ and the obvious compatibility relations), denoting by $\otimes_\nu$ the tensor product of $\mathbb{K}[[\nu]]$-modules we can identify $\tilde{A}\hat{\otimes}_\nu\tilde{A}$ with $(A\hat{\otimes}A)[[\nu]]$, where $\hat{\otimes}$ denotes the algebraic tensor product completed with respect to some topology (e.g. projective for Fréchet nuclear topology on A). We similarly have a deformed coproduct $\tilde{\Delta} = \Delta + \sum_{r=1}^{\infty} \nu^r D_r$, $D_r \in \mathscr{L}(A, A\hat{\otimes}A)$, satisfying $\tilde{\Delta}(u_1 * u_2) = \tilde{\Delta}(u_1) * \tilde{\Delta}(u_2)$.

In this context appropriate cohomologies can be introduced [GS90, Bo92]. There are natural additional requirements for Hopf algebras.

Equivalence means that there is an isomorphism $T_\nu = I + \sum_{r=1}^{\infty} \nu^r T_r$, $T_r \in \mathscr{L}(A,A)$ so that $T_\nu(u_1 *' u_2) = (T_\nu u_1 * T_\nu u_2)$ in the associative case, denoting by $*$ (resp. $*'$) the deformed laws in $\tilde{A}$ (resp. $\tilde{A}'$;) and similarly in the Lie, bialgebra and Hopf cases. In particular we see (for $r = 1$) that a deformation is trivial at order 1 if it starts with a 2-cocycle which is a 2-coboundary. More generally, exactly as above, we can show [BFFLS] ([GS88, Bo92] in the Hopf case) that if two deformations are equivalent up to some order t, the condition to extend the equivalence one step further is that a 2-cocycle (defined using the T_k, $k \le t$) is the coboundary of the required T_{t+1} and therefore *the obstructions to equivalence lie in the 2-cohomology*. In particular, if that space is null, all deformations are trivial.

Unit. An important property is that a *deformation of an associative algebra with unit* (what is called a unital algebra) is again unital, and *equivalent to a deformation with the same unit*. This follows from a more general result of Gerstenhaber (for deformations leaving unchanged a subalgebra) and a proof can be found in [GS88].

REMARK 1. In the case of (topological) *bialgebras* or *Hopf* algebras, *equivalence* of deformations has to be understood as an isomorphism of (topological) $\mathbb{K}[[\nu]]$-algebras, the isomorphism starting with the identity for the degree 0 in ν. A deformation is again said to be *trivial* if it is equivalent to that obtained by base field extension. For Hopf algebras the deformed algebras may be taken (by equivalence) to have the same unit and counit, but in general not the same antipode.

II.2 Complements

II.2.1 A brief historical survey; contractions and deformations

The discovery, about 2000 years ago, of the non-flat nature of Earth is probably the first empirical introduction of the notion of deformation in our description of the universe. Closer to us, the paradox coming from the Michelson and Morley experiment (1887) was resolved in 1905 by Einstein with the special theory of relativity: in our context, one can express that by saying that the Galilean geometrical symmetry group of Newtonian mechanics is deformed to the Poincaré group, the deformation parameter being c^{-1} where c is the velocity of light in vacuum.

Curiously, it is the (less precisely defined) inverse notion of *contraction* of symmetries that was first introduced in mathematical physics [Se51, WI53, Sa61]. Contractions, "limits of Lie algebras" as they were called in the first examples, can be viewed as an inverse of deformations – but not necessarily of Gerstenhaber-type deformations (see 2.2.3 below). We shall not expand here on that "inverse" notion but, for completeness, give its flavor. A (finite dimensional) Lie algebra can be described in a given basis L_i ($i = 1,\ldots,n$) by its structure constants $C^k_{i,j}$. The equations governing the skew symmetry of the Lie bracket and the Jacobi identity insure that the set of all structure constants lies on an algebraic variety in that n^3 dimensional space [Ln67]. A contraction is obtained e.g. when one makes a simple basis change of the form $L'_i = \varepsilon L_i$ on some of the basis elements, and then lets $\varepsilon \to 0$. Take for example $n = 3$ and restrict to the 3-dimensional subspace of the algebraic variety of 3-dimensional Lie algebras with commutation relations $[L_1, L_2] = c_3 L_3$ and cyclic permutations. The semi-simple algebras $\mathfrak{so}(3)$ and $\mathfrak{so}(2,1)$ are obtained in the open set $c_1c_2c_3 \neq 0$. A contraction gives the Euclidean algebras, where one c_i is 0. The "coordinate axes" (two of the c_i's are 0) give the Heisenberg algebra and the origin is the Abelian Lie algebra. This is of course a partial picture (e.g. solvable algebras are missing) but it is characteristic. The passage from the Poincaré Lie algebra to the Galilean is a higher dimensional version of it. Dirac constraints (mathematically, a restriction from $\mathbb{R}^{2n}$ to a symplectic or Poisson submanifold, see e.g. [Li75]) can give such contractions and be interpreted in terms of star products [AC79].

An implicit mathematical example of deformations (in a geometric context) was introduced in mid-19^{th} century by Riemann who counted the number of 'moduli' or parameters of Riemann surfaces. Teichmüller[1] [Tm39] made deformations of Riemann surfaces explicit and identified infinitesimal deformations as quadratic differentials. The correct definition, applicable to complex manifolds of arbitrary dimension, is in the short but ground breaking note of Frölicher and Nijenhuis [FN57]. They showed that if all infinitesimal deformations vanished then the manifold was rigid ("stable" in their terminology), i.e., possessed no global deformations. This was the impetus for the deep and comprehensive work of Kodaira and Spencer [KS58]. Curiously, however, the possibility of obstructions to infinitesimal deformations (which can not occur in the case of Riemann surfaces because of the low dimension) was not originally understood, and appeared in the words of Kodaira and Spencer as an "experimental fact".

Now, when one has an action on a geometrical structure, it is natural to try and "linearize" it by inducing from it an action on an algebra of functions on that structure. That is implicitly what Gerstenhaber did in [Ge64] with his definition and thorough study of deformations of rings and algebras. We shall encounter the concept of contraction more explicitly as it relates to quantum groups later in Section VI.4

II.2.2 Homotopy of deformations

For reasons that are related to the so-called Donald–Flanigan conjecture, two of us considered (see [GG98b, GGSp]) the question of (formal) *compatibility* of deformations, a kind of homotopy in the variety of algebras between two deformations (1) with parameters ν and ν' and cochains C_r and C'_r. By this he means a 2–parameter deformation of the form

$$u_1 \tilde{*} u_2 = u_1 u_2 + \nu C_1(u_1,u_2) + \nu' C'_1(u_1,u_2) + \sum_{r=2}^{\infty} \Phi_r(u_1,u_2;\nu,\nu') \tag{3}$$

where each Φ_r is a polynomial of total degree r in ν and ν', which reduces to the first one-parameter deformation when $\nu' = 0$ and to the second when $\nu = 0$. At the first order the condition for this to hold (e.g. for associative algebras) is that the Gerstenhaber bracket [Ge64] $[C_1, C'_1]_G$ is a 3-coboundary, and here also there are higher obstructions. As an example, it follows from [HKR] that the Weyl algebra and the quantum plane are formally (but non analytically [GZ95]) compatible nonequivalent deformations of the polynomial algebra $\mathbb{C}[x,y]$. Below we shall see another appearance of such a 2-parameter deformation in a physical context [BFLS].

II.2.3 More general deformations.

Deformations that are more general than the "DrG-deformations" of Gerstenhaber can (and have been) introduced, where e.g. the deformation "parameter" may act on the algebra.

In 1973 Nambu [Nb73] published some calculations which he had made a dozen years before: with quarks in the back of his mind he started with a kind of "Hamilton equations" on $\mathbb{R}^3$ with two "Hamiltonians" g, h functions of r. In this new mechanics the evolution of a function f on $\mathbb{R}^3$ is $\frac{df}{dt} = \frac{\partial(f,g,h)}{\partial(x,y,z)}$, a 3-bracket, where the right-hand side is the Jacobian of the mapping $\mathbb{R}^3 \to \mathbb{R}^3$ given by $(x,y,z) \mapsto (f,g,h)$. That expression was easily generalized, e.g. to n functions f_i, $i = 1,\dots,n$.

In order to quantize the Nambu bracket a natural idea is to replace, in the definition of the Jacobian, the pointwise product of functions by a deformed product. For this to make sense, the deformed product should be Abelian, so we are lead to consider commutative DrG-deformations of an associative and commutative product. But the commutative part of Hochschild cohomology (called Harrison cohomology) is trivial, at least in the absence of singularities (see however [Fr02]). So in [DFST] a kind of "second

[1] "It does not of necessity follow that, if the work delights you with its grace, the one who wrought it is worthy of your esteem", *cf.* Lipman Bers [Bers] (p. 324, after Plutarch, Pericles 2.1; Lives, Loeb's Classical Library **3**, p.5) who however, with André Weil, despite personal abhorrence of someone who helped drive Jewish mathematicians from Nazi Germany, fully credited his work.

quantization" procedure was used, where the deformation parameter behaves as if it was nilpotent (like Pauli matrices).

This triggered Pinczon [Pi97] and Nadaud [Na98] to generalize the Gerstenhaber theory to the case of a deformation parameter which *does not commute with the algebra*, but acts on it. Though this generalization of deformations does not (yet) give Nambu mechanics quantization, it opens a whole new direction of research for deformation theory. In particular [Pi97], while the Weyl algebra W_1 (generated by the Heisenberg Lie algebra $\mathfrak{h}_1$) is known to be DrG-rigid, it can be nontrivially deformed in such a *supersymmetric deformation theory* to the supersymmetry enveloping algebra $\mathscr{U}(\mathfrak{osp}(1,2))$. Also [Na98] on the polynomial algebra $\mathbb{C}[x,y]$ in 2 variables, Moyal-like products of a new type were discovered. All these deformations give the original algebra by a contraction, when the parameter goes to 0. So there is life outside the DrG framework, even if that is so far largely unexplored.

III QUANTUM MECHANICS AS A DEFORMATION

III.1 The setting

Intuitively, classical mechanics is the limit of quantum mechanics when $\hbar = \frac{h}{2\pi}$ goes to zero. But how can this be realized when in classical mechanics the observables are functions over phase space (a Poisson manifold) and not operators? The deformation philosophy promoted by Flato shows the way: one has to look for deformations of algebras of classical observables, functions over Poisson manifolds, and realize there quantum mechanics in an *autonomous* manner.

What we call "deformation quantization" relates to (and generalizes) what in the conventional (operatorial) formulation are the Heisenberg picture and Weyl's quantization procedure. In the latter [We31], starting with a classical observable $u(p,q)$, some function on phase space $\mathbb{R}^{2\ell}$ (with $p,q \in \mathbb{R}^{\ell}$), one associates an operator (the corresponding quantum observable) $\Omega(u)$ in the Hilbert space $L^2(\mathbb{R}^{\ell})$ by the following general recipe:

$$u \mapsto \Omega_w(u) = \int_{\mathbb{R}^{2\ell}} \tilde{u}(\xi,\eta) \exp(i(P.\xi + Q.\eta)/\hbar) w(\xi,\eta)\, d^{\ell}\xi d^{\ell}\eta \tag{4}$$

where $\tilde{u}$ is the inverse Fourier transform of u, P_α and Q_α are operators satisfying the canonical commutation relations $[P_\alpha, Q_\beta] = i\hbar\delta_{\alpha\beta}$ $(\alpha,\beta = 1,...,\ell)$, w is a weight function and the integral is taken in the weak operator topology. What is called in physics normal (or antinormal) ordering corresponds to choosing for weight $w(\xi,\eta) = \exp(-\frac{1}{4}(\xi^2 \pm \eta^2))$. Standard ordering (the case of the usual pseudodifferential operators in mathematics) corresponds to $w(\xi,\eta) = \exp(-\frac{i}{2}\xi\eta)$ and the original Weyl (symmetric) ordering to $w = 1$. An inverse formula was found shortly afterwards by Eugene Wigner [Wi32] and maps an operator into what mathematicians call its symbol by a kind of trace formula. For example Ω_1 defines an isomorphism of Hilbert spaces between $L^2(\mathbb{R}^{2\ell})$ and Hilbert-Schmidt operators on $L^2(\mathbb{R}^{\ell})$ with inverse given by

$$u = (2\pi\hbar)^{-\ell}\, \mathrm{Tr}[\Omega_1(u) \exp((\xi.P + \eta.Q)/i\hbar)] \tag{5}$$

and if $\Omega_1(u)$ is of trace class one has $\mathrm{Tr}(\Omega_1(u)) = (2\pi\hbar)^{-\ell} \int u\,\omega^{\ell} \equiv \mathrm{Tr_M}(u)$, the "Moyal trace", where ω^{ℓ} is the (symplectic) volume dx on $\mathbb{R}^{2\ell}$. Looking for a direct expression for the symbol of a quantum commutator, Moyal found [Mo49] what is now called the Moyal bracket:

$$M(u_1,u_2) = \nu^{-1}\sinh(\nu P)(u_1,u_2) = P(u_1,u_2) + \sum_{r=1}^{\infty} \frac{\nu^{2r}}{(2r+1)!} P^{2r+1}(u_1,u_2) \tag{6}$$

where $2\nu = i\hbar$, $P^r(u_1,u_2) = \Lambda^{i_1 j_1} \dots \Lambda^{i_r j_r}(\partial_{i_1 \dots i_r} u_1)(\partial_{j_1 \dots j_r} u_2)$ is the r^{th} power $(r \geq 1)$ of the Poisson bracket bidifferential operator P, $i_k, j_k = 1,\dots,2\ell$, $k = 1,\dots,r$ and $(\Lambda^{i_k j_k}) = \binom{0\,-I}{I\;0}$. To fix ideas we may assume here $u_1,u_2 \in \mathscr{C}^\infty(\mathbb{R}^{2\ell})$ and the sum is taken as a formal series. A corresponding formula for the symbol of a product $\Omega_1(u)\Omega_1(v)$ can be found in [Gr46], and may now be written more clearly as a (Moyal) *star product*:

$$u_1 *_M u_2 = \exp(\nu P)(u_1,u_2) = u_1u_2 + \sum_{r=1}^{\infty} \frac{\nu^r}{r!} P^r(u_1,u_2). \tag{7}$$

The formal series may be deduced (see e.g. [Bi00]) from an integral formula of the type:

$$(u_1 * u_2)(x) = c_\hbar \int_{\mathbb{R}^{2\ell}\times\mathbb{R}^{2\ell}} u_1(x+y)u_2(x+z)e^{-\frac{i}{\hbar}\Lambda^{-1}(y,z)}dydz. \tag{8}$$

Other integral formulas are known for quite some time (see e.g. [Ma86] where the Weyl correspondence between bounded operators in $L^2(\mathbb{R}^l)$ and bounded twisted convolution operators of $L^2(\mathbb{R}^{2l})$ is also described). It was noticed, however after deformation quantization was introduced, that the composition of symbols of pseudodifferential operators (ordered, like differential operators, “first q, then p”) is a star product.

One recognizes in (7) a special case of (1), and similarly for the bracket. So, via a Weyl quantization map, the algebra of quantized observables can be viewed as a deformation of that of classical observables.

But the deformation philosophy tells us more. Deformation quantization is not merely “a reformulation of quantizing a mechanical system” [DN01], e.g. in the framework of Weyl quantization: *The process of quantization itself is a deformation.* In order to show that explicitly it was necessary to treat in an *autonomous* manner significant physical examples (in effect, those for which a complete and rigorous spectral theory exists,) without recourse to the traditional operatorial formulation of quantum mechanics. That was achieved in [BFFLS] with the paradigm of the harmonic oscillator and more, including the angular momentum and the hydrogen atom.

In particular what plays here the role of the unitary time evolution operator of a quantized system is the “star exponential” of its classical Hamiltonian H (expressed as a usual exponential series but with “star powers” of $tH/i\hbar$, t being the time, and computed as a distribution both in phase space variables and in time); in a very natural manner, the spectrum of the quantum operator corresponding to H is the support of the Fourier-Stieltjes transform (in t) of the star exponential (what Laurent Schwartz had called the spectrum of that distribution). It is worth mentioning that our definition of spectrum permits to define a spectrum even for symbols of non-spectrable operators, such as the derivative on a half-line which has different deficiency indices; this corresponds to an infinite potential barrier. That is one of the many advantages of our autonomous approach to quantization. Further examples were (and are still being) developed, in particular in the direction of field theory.

III.2 Quantum mechanics without operators: Harmonic oscillator, angular momentum and hydrogen atom.

In quantum mechanics it is preferable to work (for $X=\mathbb{R}^{2\ell}$) with the Moyal product, which has maximal symmetry, i.e. has $\mathfrak{sp}(\mathbb{R}^{2\ell})\cdot\mathfrak{h}_\ell$ as what we call (see IV.3.2 below) algebra of preferred observables. One indeed finds [BFFLS] that star powers of these preferred observables H (polynomials of order ≤ 2) are usual polynomials in H (not only in p and q), and as a consequence their star exponential is proportional to the usual exponential and a function of H. More precisely, if $H=\alpha p^2+\beta pq+\gamma q^2\in\mathfrak{sl}(2)$ with $p,q\in\mathbb{R}^\ell$, $\alpha,\beta,\gamma\in\mathbb{R}$, setting $d=\alpha\gamma-\beta^2$ and $\delta=|d|^{1/2}$ one gets by summing the star exponential (with deformation parameter $\nu=\frac{i\hbar}{2}$) and then taking its Fourier (or Fourier-Stieltjes) development, the sums and integrals appearing in the various expressions of the star exponential being convergent as distributions, in phase-space variables and in t or λ:

$$\mathrm{Exp}(Ht) = \begin{cases} (\cos\delta t)^{-l}\exp((H/i\hbar\delta)\tan(\delta t)) & \text{for} \quad d>0 \\ \exp(Ht/i\hbar) & \text{for} \quad d=0 \\ (\cosh\delta t)^{-l}\exp((H/i\hbar\delta)\tanh(\delta t)) & \text{for} \quad d<0 \end{cases} \tag{9}$$

$$\mathrm{Exp}(Ht) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} \Pi_n^{(\ell)} e^{(n+\frac{\ell}{2})t} & \text{for} \quad d > 0 \\ \displaystyle\int_{-\infty}^{\infty} e^{\lambda t/i\hbar} \Pi(\lambda, H) d\lambda & \text{for} \quad d < 0 \end{cases} \tag{10}$$

We thus get the discrete spectrum $(n + \frac{\ell}{2})\hbar$ of the *harmonic oscillator* and the continuous spectrum $\mathbb{R}$ for the dilation generator pq. The eigenprojectors $\Pi_n^{(\ell)}$ and $\Pi(\lambda, H)$ are given [BFFLS] by known special functions on phase-space (generalized Laguerre and hypergeometric, multiplied by some exponential). Formulas (9) and (10) can, by analytic continuation, be given a sense outside singularities and even (as distributions) for values of t for which the expressions are singular.

Other orderings give similar formulas [Ma04] and other examples can be brought to this case, in particular by functional manipulations [BFFLS]. For instance the Casimir element C of $\mathfrak{so}(\ell)$ representing *angular momentum*, which can be written $C = p^2q^2 - (pq)^2 - \ell(\ell-1)\frac{\hbar^2}{4}$, has $n(n+(\ell-2))\hbar^2$ for spectrum. For the *hydrogen atom*, with Hamiltonian $H = \frac{1}{2}p^2 - |q|^{-1}$, the Moyal product on $\mathbb{R}^{2\ell+2}$ ($\ell = 3$ in the physical case) induces a star product on $X = T^*S^\ell$; the energy levels, solutions of $(H-E) * \phi = 0$, are found from (10) and the preceding calculations for angular momentum to be (as they should, with $\ell = 3$) $E = \frac{1}{2}(n+1)^{-2}\hbar^{-2}$ for the discrete spectrum, and $E \in \mathbb{R}^+$ for the continuous spectrum.

We thus have recovered, in a completely autonomous manner entirely within deformation quantization, the results of “conventional” quantum mechanics in these typical examples (and many more can be treated similarly). It is worth noting that the term $\frac{\ell}{2}$ in the harmonic oscillator spectrum, obvious source of divergences in the infinite-dimensional case, disappears if the normal star product is used instead of Moyal – which is one of the reasons it is preferred in field theory.

III.3 Modern developments

Since the original papers in 1976–78 [FLS, BFFLS], deformation quantization has been extended considerably. It now includes general symplectic and Poisson (finite dimensional) manifolds, with further results for infinite dimensional manifolds, for “manifolds with singularities” and for algebraic varieties, and has many far reaching ramifications in both mathematics and physics (see e.g. a brief overview in [DS02]). As in quantization itself [We31], symmetries (group theory) play a special role and an autonomous theory of star representations of Lie groups was developed, in the nilpotent and solvable cases of course (due to the importance of the orbit method there), but also in significant other examples. The presentation that follows can be seen as an extension of the latter, when one makes full use of the Hopf algebra structures and of the “duality” between the group structure and the set of its irreducible representations.

It goes without saying (but mentioning it won’t hurt) that deformation theory and Hopf algebras are seminal in a variety of problems ranging from theoretical physics to algebraic geometry, number theory and more. We shall not insist here on the manifold applications of quantum groups, certainly treated in many contributions to this special issue. In theoretical physics one finds now applications (see e.g. [CK99, DS02, Ta03]) to renormalization and Feynman integrals and diagrams. Noncommutativity is a staple in modern theoretical physics [DN01] (including string theory and its avatars) even at the level of space-time; deformation quantization (in particular Moyal products) is an important tool there, at least at the formal level. But the applications (and by-products therefrom to physics) extend as far as algebraic geometry and number theory (see e.g. [Ko01, KZ01]), including algebraic curves à la Zagier (cf. [CM03, CM04] and Connes’ lectures at Collège de France, Winter 2003 and 2004).

IV QUANTUM GROUPS AS DEFORMATIONS

In this section we explore in detail the concept of viewing quantum groups as Hopf algebra deformations of $\mathscr{U}\,\mathfrak{g}$ and $\mathscr{C}^\infty(G)$. We restrict to the case where G is a semi-simple Lie group with Lie algebra $\mathfrak{g}$.

IV.1 Formal Deformations

We begin with a summary of important considerations regarding *formal deformations*: all vector spaces, tensor products, etc. being complete in the ν-adic topology. This is a purely algebraic approach with no consideration to any topological or convergence questions. All of what follows can be extracted from [D89a, D89b, GS90, GS90b].

Recall that a bialgebra is called *rigid* it every deformation is trivial. Neither $\mathscr{U}\,\mathfrak{g}$ nor $\mathscr{C}^\infty(G)$ is rigid but, as the theorem below asserts, each is *half-rigid* in the following sense: $\mathscr{U}\,\mathfrak{g}$ is rigid as an algebra and $\mathscr{C}^\infty(G)$ is rigid as a coalgebra. This means that for any deformation $\mathscr{U}_\nu(\mathfrak{g})$, there is an equivalent one in which the original multiplication $\mu_0 : \mathscr{U}\,\mathfrak{g}\otimes\mathscr{U}\,\mathfrak{g}\to\mathscr{U}\,\mathfrak{g}$ is preserved and so $\mathscr{U}_\nu(\mathfrak{g})\cong(\mathscr{U}\,\mathfrak{g}[[\nu]],\mu_0,\tilde{\Delta})$ for some coassociative $\tilde{\Delta}$. Similarly, any deformation $\mathscr{C}^\infty_\nu(G)$ is equivalent to one in which the comultiplication $\Delta_0 : \mathscr{C}^\infty(G)\to\mathscr{C}^\infty(G)$ is preserved on all elements. Thus $\mathscr{C}^\infty_\nu(G)\cong(\mathscr{C}^\infty(G)[[\nu]],*,\Delta_0)$ for some associative $*$-product. Such deformations in which one of the original structure maps is preserved are called *preferred.*

The structure of the deformed comultiplication of $\mathscr{U}\,\mathfrak{g}$ and the deformed multiplication $\mathscr{C}^\infty(G)$ are produced via certain elements of $(\mathscr{U}\,\mathfrak{g}\otimes\mathscr{U}\,\mathfrak{g})[[\nu]]$. For $F\in(\mathscr{U}\,\mathfrak{g}\otimes\mathscr{U}\,\mathfrak{g})[[\nu]]$ and $u\in\mathscr{U}\,\mathfrak{g}$, define $\Delta_F(u) = F\Delta_0(u)F^{-1}$. In a dual sense, we can define $f*_F g$ for $f,g\in\mathscr{C}^\infty(G)$. To define $*_F$ we need some notation first. For $x\in\mathfrak{g}$, let x_λ and x_ρ be the left invariant and right invariant, respectively, derivations of $\mathscr{C}^\infty(G)$ arising from the corresponding left and right invariant vector fields on G associated to x. Taking the tensor product and extending linearly, we associate to every $F\in(\mathscr{U}\,\mathfrak{g}\otimes\mathscr{U}\,\mathfrak{g})[[\nu]]$ two formal sums of bidifferential operators F_λ and F_ρ. With this, we define $f*_F g = \mu_0\circ(F_\lambda\circ F_\rho^{-1})(f\otimes g)$. Note that if $F\cong 1\otimes 1 \mod \nu$, then $\Delta_F(u)$ and $f*_F g$ are series whose constant terms are the original structure maps $\Delta_0(u)$ and $f\cdot g$. However, Δ_F will not generally be coassociative and $*_F$ will not generally be associative. The appropriate condition for F to satisfy is given by the following important result.

Theorem IV.1 ([D89a, D89b, GS90b]) *Let G be a semi-simple Lie group with Lie algebra $\mathfrak{g}$, and suppose that $F\in(\mathscr{U}\,\mathfrak{g}\otimes\mathscr{U}\,\mathfrak{g})[[\nu]]$ with $F\cong 1\otimes 1 \mod \nu$.*
Then $(\mathscr{U}\,\mathfrak{g}[[\nu]],\mu_0,\Delta_F)$ and $(\mathscr{C}^\infty(G)[[\nu]],*_F,\Delta_0)$ are deformations if and only if

$$F_{12}(\Delta_0\otimes 1)F = \Phi F_{23}(1\otimes\Delta_0)F \tag{11}$$

for some $\Phi\in(\mathscr{U}\,\mathfrak{g}\otimes\mathscr{U}\,\mathfrak{g}\otimes\mathscr{U}\,\mathfrak{g})^{\mathfrak{g}}$. Moreover, every deformation of $\mathscr{U}\,\mathfrak{g}$ and $\mathscr{C}^\infty(G)$ is equivalent to one defined by such an F.

Note that (11) is an equation which must hold in the triple tensor product $\mathscr{U}\,\mathfrak{g}\otimes\mathscr{U}\,\mathfrak{g}\otimes\mathscr{U}\,\mathfrak{g}$. (The comultiplication Δ_0 in this equation is that of $\mathscr{U}\,\mathfrak{g}$.) The half-rigidity of $\mathscr{U}\,\mathfrak{g}$ was first proved in [D89a] and it was derived for $\mathscr{C}^\infty(G)$ in [GS90b]. The specific form of the deformations can be deduced from [D89b].

If $\Phi = 1\otimes 1\otimes 1$ (a trivial $\mathfrak{g}$-invariant), then F is called a *twisting element* and, if Φ is a non-trivial invariant then F is a *modified twisting element.* In either case, we say the deformation is given by a "twist."

From the viewpoint of deformation quantization, it may seem that Theorem IV.1 settles the story. In a way it does, but it opens up many more questions which we will address throughout the remainder of this survey. The most basic question is to find elements F which solve Equation (11). This task is easier said than done. Indeed, no modified twisting elements are explicitly known for any simple Lie algebra $\mathfrak{g}$

– even for the rank 1 case of $\mathfrak{sl}(2)$! The situation is different for twisting elements – there is a handful which are explicitly known, and we will exhibit some of them later in Section VI.2.

There is an interesting irony here concerning the infinitesimals, (or first order terms) of the two types of twisting elements. The possible infinitesimals of modified twisting elements are all constructively classified in the famous paper [BD84] of Belavin and Drinfel′d. In contrast, such a classification for the infinitesimals of the twisting elements is an intractable problem. (It would require, in particular, a constructive classification of all abelian subalgebras of $\mathfrak{g}$, a question which is known to be too broad to solve as stated.)

IV.2 Topological Preliminaries

In the next few sections, we describe a "topological" approach to quantum groups via deformation quantization. Instead of working formally over complete power series rings, we consider other topologies which have more desirable properties, especially in terms of dualization. The theory was initiated in [BFGP] and its generalizations can be found in [BP96].

IV.2.1 Example: the $SU(2)$ case

First we will investigate in detail the example that the general theory is based on. It is interesting and merits special attention because apart from giving the main ideas, it differs in two crucial points: it is *explicit* and *convergent.* The main reference is [BFP].

We start with Jimbo's definition of $\mathscr{U}_q\mathfrak{sl}(2)$ [Ji85]. As an algebra, $\mathscr{U}_q\mathfrak{sl}(2)$ has generators E, F, $K^{\pm 1}$ and the relations

$$KEK^{-1} = qE, \quad KFK^{-1} = q^{-1}F, \quad EF - FE = \frac{K^2 - K^{-2}}{q - q^{-1}}.$$

The coalgebra structure on $\mathscr{U}_q\mathfrak{sl}(2)$ is given by

$$\Delta(E) = E \otimes K^{-1} + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + K \otimes F, \quad \Delta(K) = K \otimes K.$$

Remark IV.1

1. *One may view $\mathscr{U}_q\mathfrak{sl}(2)$ as an algebra over $\mathbb{K}$ (with $q \in \mathbb{K}^*$) or over the rational function field $\mathbb{K}(q)$ (with q an indeterminate). For what follows it will be convenient to view $q = e^{i\nu} \in \mathbb{C}$.*
2. *Note that, as presented, $\mathscr{U}_q\mathfrak{sl}(2)$ is not a deformation of $\mathscr{U}\mathfrak{sl}(2)$. If one formally sets $K = q^{H/2}$ (where $[E,F] = H$ in $\mathfrak{sl}(2)$) and completes the algebra with respect to the ν-adic topology, then one indeed has a genuine deformation of $\mathscr{U}\mathfrak{sl}(2)$. This fact, however, is not obvious from looking at just the relations. It is usually proved by analyzing the representation theory of the algebras involved.*

Even though $\mathscr{U}_q\mathfrak{sl}(2)$ is undefined at $q = 1$, one may make sense of it as q approaches 1. Specifically, under the linear change of generators, $S = \frac{K-K^{-1}}{q-q^{-1}}$ $\ C = \frac{K+K^{-1}}{2}$, one has the following set of relations:

$$\begin{array}{ll} EF - FE = 2SC & SC = CS \\ ES = (S\cos\nu - C)E & EC = (C\cos\nu + S\sin^2\nu)E \\ FS = (S\cos\nu + C)F & FC = (C\cos\nu - S\sin^2\nu)F \\ C^2 + S^2\sin^2\nu = 1. & \end{array} \tag{12}$$

This new system of generators and relations is well-defined for all ν, and it turns out that $\mathscr{U}_1\mathfrak{sl}(2) \cong \mathscr{U}\mathfrak{sl}(2) \otimes \mathbb{K}[X]/(X^2 - 1)$. The element X is called a *parity*. The Jimbo quantum groups $\mathscr{U}_q\mathfrak{g}$ have, in general, r parities, where r is the rank of $\mathfrak{g}$ and so they are clearly not a formal deformation of $\mathscr{U}\mathfrak{g}$, notwithstanding the aforementioned rigidity result of Theorem (IV.1).

The algebra $\mathscr{A}$. Let $\{(\pi_n, V_n)\}_{n\in\frac{1}{2}\mathbb{N}}$ be the finite dimensional irreducible representations of $\mathfrak{sl}(2)$ (or $SU(2)$, as they are the same). Let $\mathscr{A} = \prod_{n\in\frac{1}{2}\mathbb{N}} \mathtt{End}(V_n)$. We consider the product topology on $\mathscr{A}$ (Fréchet). We have embeddings $\mathscr{U}\,\mathfrak{sl}(2) \hookrightarrow \mathscr{A}$ and $\mathbb{C}G \hookrightarrow \mathscr{A}$ by $u \mapsto (\pi_n(u))$ and $x \mapsto (\pi_n(x))$. These maps are injective because $\{(\pi_n, V_n)\}$ is a complete set of representations for $\mathfrak{sl}(2)$ (or $SU(2)$).

Let π be the representation of $\mathscr{U}\,\mathfrak{sl}(2)$ (or $SU(2)$) defined by $\sum_{n\in\frac{1}{2}\mathbb{N}} \pi_n$. Then $\mathscr{A}$ coincides with the bicommutant of π and, by semi-simplicity of π and the density theorem of Jacobson, we get

$$\overline{\mathscr{U}\,\mathfrak{sl}(2)} = \mathscr{A} \text{ and } \overline{\mathbb{C}G} = \mathscr{A}.$$

Since the set $\{(\pi_n, V_n)\}_{n\in\frac{1}{2}\mathbb{N}}$ is also a complete set of representations for $\mathscr{U}_q\mathscr{G}$ we have the following similar results

$$\mathscr{U}_q sl(2) \hookrightarrow \mathscr{A} \quad \text{ and } \quad \overline{\mathscr{U}_q\mathfrak{sl}(2)} = \mathscr{A}, \quad \text{ for all } \quad \nu \notin 2\pi\mathbb{Q}.$$

Details can be found in [BFP].

Thus, both $\mathscr{U}\,\mathfrak{sl}(2)$ and $\mathscr{U}_q\mathfrak{sl}(2)$ are dense subalgebras of $\mathscr{A}$. We denote by A_ν the subalgebra of $\mathscr{A}$ isomorphic to $\mathscr{U}_q sl(2)$. Since $A_\nu \cong A_{\nu'}$ if and only if $\nu' = \pm\nu + 2k\pi,\ k \in \mathbb{Z}$, it follows that A_ν is a deformation of A_0.

By a standard density argument, the coproduct Δ_ν of $A_\nu,\ \nu \notin 2\pi\mathbb{Q}$, can be extended to the whole algebra $\mathscr{A}$; the same holds true for the antipode. Thus we have a preferred deformation of $\mathscr{A}$ as the coalgebra structure varies with a fixed algebra structure. In fact, this deformation of $\mathscr{A}$ has the same form as any preferred deformation of $\mathscr{U}\,\mathfrak{g}$. Specifically, there is an invertible element $F \in \mathscr{A}\hat{\otimes}\mathscr{A}$ such that $\Delta_\nu = F\,\Delta_0\,F^{-1}$. The element F is constructed component by component using equivalences of representations, see [BFP] for details. A consequence is that all the Hopf structures on $\mathscr{A}$ induced by various A_ν are isomorphic as quasi-Hopf algebras (see [D89b]).

So far we have not used the topology on $\mathscr{A}$, but it will play a crucial role as we now want to consider its dualization. The strong topological dual of $\mathscr{A} = \prod_{n\in\frac{1}{2}\mathbb{N}} \mathtt{End}(V_n)$ is $\mathscr{A}^* = \mathscr{H} = \bigoplus_{n\in\frac{1}{2}\mathbb{N}} \mathtt{End}(V_n)$, the coefficient Hopf algebra of polynomial functions on $SU(2)$. Since $(\mathscr{A}\hat{\otimes}\mathscr{A})^* = \mathscr{H}\hat{\otimes}\mathscr{H}$, we obtain a Hopf algebra on $\mathscr{A}^*$, which we denote $\mathscr{H}_\nu$,

Proposition IV.1 *$\mathscr{H}_\nu$ coincides precisely with the deformation of the function Hopf algebra $\mathbb{C}[SL_2]$, the quantum group $\mathbb{C}[SL_2]$, given in [FRT].*

Thus we have shown that there is an explicit embedding of the Jimbo quantum group $\mathscr{U}_q\mathfrak{sl}(2),\ q \in \mathbb{C}$ ($q^n \neq 1$ for all $n \in \mathbb{N}$) in the $\mathbb{C}$-Hopf algebra $\mathscr{A}$.

In the general case that will not be possible. We shall have to use the $\mathbb{C}[[\nu]]$-Hopf algebra $\mathscr{A}[[\nu]]$ and the non-explicit Drinfel′d isomorphisms.

IV.2.2 Some topological Hopf algebras (Well-Behaved Hopf algebras)

We shall now briefly review applications of the deformation theory of algebras in the context of Hopf algebras endowed with appropriate topologies and in the spirit of deformation quantization. That is, we shall consider Hopf algebras of functions on Poisson-Lie groups (or their topological duals) and their deformations, and show how this framework is a powerful tool to understand the standard examples of quantum groups, and more. In order to do so we first recall some notions on topological vector spaces and apply them to our context.

Definition IV.1 *A topological vector space (tvs) V is said* well-behaved *if V is either nuclear and Fréchet, or nuclear and dual of Fréchet [Gr55, Tr67].*

Proposition IV.2 *If V is a well-behaved tvs and W a tvs, then*

$$(i)\ V^{**} \simeq V \qquad (ii)\ (V\hat{\otimes}V)^* \simeq V^*\hat{\otimes}V^* \qquad (iii)\ \mathtt{Hom}_{\mathbb{K}}(V,W) \simeq V^*\hat{\otimes}W$$

where V^* denotes the strong topological dual of V, $\hat{\otimes}$ the projective topological tensor product and the base field $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.

Definition IV.2 $(A,\mu,\eta,\Delta,\varepsilon,S)$ *is a WB (well-behaved) Hopf algebra [BFGP] if*

- *A is a well-behaved topological vector space.*
- *The multiplication $\mu : A\hat{\otimes}A \to A$, the coproduct $\Delta : A \to A\hat{\otimes}A$, the unit η , the counit ε , and the antipode S are continuous.*
- *$\mu,\eta,\Delta,\varepsilon$ and S satisfy the usual axioms of a Hopf algebra.*

Corollary IV.1 *If $(A,\mu,\eta,\Delta,\varepsilon,S)$ is a WB Hopf algebra, then $(A^*,{}^t\Delta,{}^t\varepsilon,{}^t\mu,{}^t\eta,{}^tS)$ is also a WB Hopf algebra.*

Examples IV.1 Let G be a semi-simple Lie group and $\mathfrak{g}$ its complexified Lie algebra.

1. $\mathscr{C}^\infty(G)$, the algebra of the smooth functions on G, is a WB Hopf algebra (Fréchet and nuclear).
2. $\mathscr{D}(G) = \mathscr{C}^\infty(G)^*$, the algebra of the compactly supported distributions on G, is a WB Hopf algebra (dual of Fréchet and nuclear). The product is the transposed map of the coproduct of $\mathscr{C}^\infty(G)$ that is, the convolution of distributions.
3. $\mathscr{H}(G)$, the algebra of coefficient functions of finite dimensional representations of G (or polynomial functions on G) is a WB Hopf algebra, the Hopf structure being that induced from $\mathscr{C}^\infty(G)$.

   A short description of that algebra is as follows: We take a set $\hat{G}$ of irreducible finite dimensional representations of G such that there is *one and only one* element for each equivalence class, and, if $\pi \in \hat{G}$, its contragredient $\check{\pi}$ is also in $\hat{G}$. We define

   $$C_\pi = \mathrm{vect}\{\text{coefficient functions of } \pi\} \overset{Burnside}{\simeq} \mathtt{End}(V_\pi) \ \text{ for } \ \pi \in \hat{G}.$$

   Then $\mathscr{H}(G) \overset{alg.}{\simeq} \bigoplus_{\pi\in\hat{G}} C_\pi \overset{v.s.}{\simeq} \bigoplus_{\pi\in\hat{G}} \mathtt{End}(V_\pi)$. So we take on $\mathscr{H}(G)$ the "direct sum" topology. Then $\mathscr{H}(G)$ is dual of Fréchet and nuclear and so is WB.
4. Define $\mathscr{A}(G)$, the algebra of "generalized distributions", by $\mathscr{A}(G) = \mathscr{H}(G)^* \overset{alg.}{\simeq} \prod_{\pi\in\hat{G}} \mathtt{End}(V_\pi)$. The (product) topology is Fréchet and nuclear, and therefore $\mathscr{A}(G)$ is WB.

Proposition IV.3 ([BP96, BFGP]) *We denote by $\mathscr{U}\mathfrak{g}$ the universal enveloping algebra of $\mathfrak{g}$ and by $\mathbb{C}G$ the group algebra of G. All the following inclusions are inclusions of Hopf algebras. $\Subset$, $\Supset$, $\Cup$, $\Cap$ mean a* dense *inclusion.*

$$\begin{array}{ccccc|c}
\mathscr{U}\mathfrak{g} & \Subset & \mathscr{A}(G) & \overset{(*)}{\Supset} & \mathbb{C}G & \mathscr{H}(G) \\
 & & \Cup(*) & & & \Cap(*) \\
\mathscr{U}\mathfrak{g} & \subset & \mathscr{D}(G) & \Supset & \mathbb{C}G & \mathscr{C}^\infty(G)
\end{array}$$

$(*)$ *is true if and only if G is linear (i.e. with a faithful finite dimensional representation).*

IV.3 Topological quantum groups

We shall now deform the preceding topological Hopf algebras and indicate how this explains various models of quantum groups. For clarity of the exposition, throughout this Section and the remainder of the paper, we shall limit to a minimum the details concerning the Hopf algebra structures other than product and coproduct. But whenever we write Hopf algebras and not only bialgebras, the relevant structures are included in the discussion and dealing with them is quite straightforward. Note that the results which pertain only to the formal aspect of the theory were already mentioned in Section IV.1.

IV.3.1 Topological quantization

If $\mathscr{U}_\nu\mathfrak{g}$ is a deformation of $\mathscr{U}\mathfrak{g}$, then an isomorphism (it is not unique!) $\varphi : \mathscr{U}_\nu\mathfrak{g} \to \mathscr{U}\mathfrak{g}[[\nu]]$ guaranteed to exist by Theorem IV.1 will be called a *Drinfel′d isomorphism*.

Theorem IV.2 ([BFGP, BP96]) *Let G be a connected semi-simple Lie group and* $\mathfrak{g}$ *be its complexified Lie algebra.*

1. *If* $\mathscr{U}_\nu\mathfrak{g}$ *is a deformation of* $\mathscr{U}\mathfrak{g}$ *(a "quantum group") then there exists* $F \in (\mathscr{U}\mathfrak{g} \otimes \mathscr{U}\mathfrak{g})[[\nu]]$ *such that* $(\mathscr{U}_\nu\mathfrak{g}, \mu_\nu, \Delta_\nu) \simeq (\mathscr{U}\mathfrak{g}[[\nu]], \mu_0, F\Delta_0 F^{-1})$.
2. $\mathscr{A}_\nu(G) := (\mathscr{A}(G)[[\nu]], \mu_0, F\cdot\Delta_0\cdot F^{-1})$ *is a Hopf deformation of* $\mathscr{A}(G)$ *and* $\mathscr{U}_\nu\mathfrak{g} \overset{\text{Hopf}}{\subset} \mathscr{A}_\nu(G)$.
3. $\mathscr{D}_\nu(G) := (\mathscr{D}(G)[[t]], \mu_0, F\cdot\Delta_0\cdot F_\nu^{-1})$ *is a Hopf deformation of* $\mathscr{D}(G)$ *and* $\mathscr{U}_\nu\mathfrak{g} \overset{\text{Hopf}}{\subset} \mathscr{D}_\nu(G)$.
4. $\mathscr{C}^\infty_\nu(G) := \mathscr{D}_\nu(G)^*$ *and* $\mathscr{H}_\nu(G) := \mathscr{A}_\nu(G)^*$ *are quantized algebras of functions. They are Hopf deformations of* $\mathscr{C}^\infty(G)$ *and* $\mathscr{H}(G)$.

Similar results hold for other WB Hopf algebras (e.g. constructed with infinite dimensional representations) [Bg96].

Proof. Linear case: Item (1) is a direct consequence of Theorem IV.1. To prove item (2), observe that $F \in (\mathscr{U}\mathfrak{g} \otimes \mathscr{U}\mathfrak{g})[[\nu]] \subset (\mathscr{A}(G)\hat{\otimes}\mathscr{A}(G))[[\nu]]$ and coassociativity follows from the dense inclusion $\mathscr{U}\mathfrak{g} \Subset \mathscr{A}(G)$. Item (3) it true by restriction of (2), and item (4) holds by simple dualization from (2) and (3).

Non linear case: Here, as $\mathscr{D}(G) \not\subset \mathscr{A}(G)$ we have to treat $\mathscr{D}(G)$ and $\mathscr{A}(G)$ separately.
– Since we can prove (see [BP96]) that "there exists a compact connected Lie group K such that $\mathscr{H}(G) = \mathscr{H}(K)$," we have $\mathscr{A}(G) = \mathscr{A}(K)$ and we can apply the linear case.
– To treat $\mathscr{D}(G)$ we use the density of $\mathbb{C}G$ in $\mathscr{D}(G)$ and go from $\mathscr{U}\mathfrak{g}$ to $\mathbb{C}G$ by exponentiating [BP96]. ■

Remark IV.2 *"Hidden group structure" in a quantum group.* All the deformations constructed here are *preferred*, that is, the product on $\mathscr{D}_\nu(G)$ and on $\mathscr{A}_\nu(G)$ (resp. the coproduct on $\mathscr{C}^\infty_\nu(G)$ and on $\mathscr{H}_\nu(G)$) is not deformed and the basic structure is still the product on the group G. So this approach gives an interpretation of the Tannaka-Krein philosophy in the case of quantum groups: it has often been noticed that, in the generic case, finite dimensional representations of a quantum group are (essentially) representations of its classical limit. So the algebras involved should be the same, which is justified by the above mentioned rigidity result of Drinfel′d. This shows that the initial classical group is still there, acting as a kind of "hidden variables" in this quantum group theory, which is exactly what we see in this quantum group theory. This fact was implicit in Drinfel′d's work. The Tannaka-Krein interpretation of the twisting of quasi-Hopf algebras can be found in Majid (see e.g. [Mj92]). It was made explicit, within the framework exposed here, in [BFGP].

Thus, for any connected Lie group G and for any deformation of the universal enveloping algebra of $\mathfrak{g} = Lie_{\mathbb{C}}(G)$, we obtain a star product $*$ on $\mathscr{C}^\infty(G)$ and $\mathscr{H}(G)$. The next result shows that these deformation quantizations induce other ones on some quotients of G :

Proposition IV.4 *Let H be a closed normal subgroup of G.*

1. *$*$ induces a star product on $\mathscr{C}^\infty(G/H)$.*
2. *If G is linear, $*$ induces a star product on $\mathscr{H}(G/H)$.*

IV.3.2 Unification of models and generalizations

Drinfel′d models We call "Drinfel′d model of quantum group" a deformation of $\mathscr{U}\mathfrak{g}$ for $\mathfrak{g}$ simple, as given in [Dr87]. We have seen in the preceding section that from any Drinfel′d model $\mathscr{U}_\nu\mathfrak{g}$ of a quantum group (which can be generalized to any deformation of the Hopf algebra $\mathscr{U}\mathfrak{g}$), we obtain a deformation of $\mathscr{D}(G)$ and $\mathscr{A}(G)$ that contains $\mathscr{U}_\nu\mathfrak{g}$ as a sub-Hopf algebra. So $\mathscr{D}_\nu(G)$ and $\mathscr{A}_\nu(G)$ are quantum group models that describe Drinfel′d models. By duality, $\mathscr{C}_\nu^\infty(G)$ and $\mathscr{H}_\nu(G)$ are "quantum group deformations" of $\mathscr{C}^\infty(G)$ and $\mathscr{H}(G)$. The deformed product on $\mathscr{H}(G)$ is the restriction of that on $\mathscr{C}^\infty(G)$. Furthermore, as we shall see, these deformations coincide with the usual "quantum algebras of functions". Let us look more in detail at $\mathscr{H}_\nu(G)$:

Faddeev-Reshetikhin-Takhtajan (FRT) models In [FRT] quantized algebras of functions are defined in terms of generators and relations, the key relation being given by the star-triangle (Yang-Baxter) equation, $R(T\otimes \mathtt{Id})(\mathtt{Id}\otimes T) = (\mathtt{Id}\otimes T)(T\otimes \mathtt{Id})R$, for a given R-matrix $R \in \mathtt{End}(V\otimes V)$ and for $T \in \mathtt{End}(V)$, V being a finite dimensional vector space.

As our deformations are given by a twist F, it is not surprising, from a structural point of view [Mj92] that, dually, we obtain in each case a Yang-Baxter relation and so a "FRT-type" quantized algebra of functions. Our Fréchet-topological context permits to write precisely such a construction for the infinite-dimensional Hopf algebras involved.

Linear case If G is semi-simple and linear, there exists π a finite dimensional representation of G such that $\mathscr{H}(G) \simeq \mathbb{C}[\pi_{ij}; 1 \leqslant i,j \leqslant N]$ where the π_{ij} are the coefficient functions of π. Denote by $(\mathscr{H}_\nu(G), *)$ the deformation of $\mathscr{H}(G)$ obtained in this way and by T the matrix $[\pi_{ij}]$. Define $T_1 := T\otimes Id$ and $T_2 := Id \otimes T$. Then we have

Proposition IV.5 ([BFGP, BP96])

1. *$\{\pi_{ij}; 1 \leqslant i,j \leqslant N\}$is a topological generator system of the $\mathbb{C}[[\nu]]$-algebra $\mathscr{H}(G)_\nu$.*
2. *There exists an invertible $\mathscr{R} \in \mathscr{L}(V_\pi \otimes V_\pi)[[t]]$ such that $\mathscr{R}\cdot T_1 * T_2 = T_2 * T_1 \cdot \mathscr{R}$ (so $\mathscr{H}_\nu(G)$ is a "quantum algebra of functions" of type FRT).*
3. *We recover every quantum group given in [FRT] by this construction.*

Sketch of proof.

1. Perform a precise study of the deformed tensor product of representations.
2. Since the deformations $\mathscr{A}_\nu(G)$ are given by a twist F, $\mathscr{A}_\nu(G)$ is quasi-cocommutative, i.e. there exists $R \in (\mathscr{A}(G)\hat{\otimes}\mathscr{A}(G))[[\nu]]$ such that $\sigma\circ\Delta_\nu(a) = R\Delta_F(a)R^{-1}$ with $\sigma(a\otimes b) = b\otimes a$. Standard computations give the result.

3. We want to follow the ideas used in [Dr87] to link Drinfel′d to FRT models. Since the main point here is that our deformations are obtained through a Drinfel′d isomorphism, we therefore have to show:

   – There exists a specific Drinfel′d isomorphism deforming the standard representation of $\mathfrak{g}$ into the representation of $\mathscr{U}_\nu\mathfrak{g}$ used in [Dr87].

   – Two Drinfel′d isomorphisms give equivalent deformations. ∎

For instance, the FRT quantization of $SL(n)$ can be seen as a Hopf deformation of $\mathscr{H}(SU(n))$ (with non deformed coproduct). Moreover, this Hopf deformation extends to $\mathscr{C}^\infty(G)$.

Remark IV.3

1. This proposition justifies the terminology "deformation", often employed but never justified in these cases. See e.g. [GGS90] where it is shown that relations of type $\mathscr{R}T_1T_2 = T_2T_1\mathscr{R}$ need not define a deformation, even if $\mathscr{R}$ is Yang-Baxter.

2. Starting from Drinfel′d models, our construction produces FRT models also for e.g. $G = Spin(n)$ and for exceptional Lie groups. In addition, at least some multiparameter deformations [Re90] can be easily treated in this way [BFGP].

Non-linear case

Proposition IV.6 ([BP96]) *If G is semi-simple with finite center, there exists a dense subalgebra of $(\mathscr{C}^\infty_\nu(G), *)$ generated by the coefficient functions of a finite number of (possibly infinite dimensional) representations.*

Jimbo-type models The Jimbo models [Ji85] have generators $E_i^\pm$, K_i and K_i^{-1}. As stated earlier (Section IV.2.1) these are not deformations in our sense due to the presence of parities.

The $G = SU(2)$ case was developed in Section IV.2.1. Similarly, for $G = SL(2,\mathbb{C})$, Martin and Zouagui [MZ96] realize $\mathscr{U}_\nu\mathfrak{sl}(2,\mathbb{C})$ as a dense sub-Hopf algebra of $\mathscr{A}(G)$, $\forall t \in \mathbb{C}\setminus 2\pi\mathbb{Q}$ (with $q = e^\nu$). Then, for the Lorentz algebra $\mathfrak{sl}(2,\mathbb{C})$, this unifies [MZ96] all the models proposed so far in the literature for a quantum Lorentz group. We obtain here *convergent* deformations as in the above example of the $SU(2)$ case.

For $\mathfrak{sl}(2,\mathbb{C})$ it was first proposed in [PW90] to consider the quantum double [Dr87] of $\mathscr{U}_q\mathfrak{su}(2)$ as q-deformed Lorentz group. It was known from [RSts] that in such cases the double, as an algebra, is the tensor product of two copies of $\mathscr{U}_\nu\mathfrak{su}(2)$. See also [OSWZ, SWZ], and [Mj93] for a dual version and another semi-direct product form.

Deformation quantization From the main construction, using deformations of $\mathscr{U}\mathfrak{g}$, we deduce the following general theorem:

Theorem IV.3 ([BP96]) *Let G be a semi-simple connected Lie group with a Poisson-Lie structure. There exists a deformation $(\mathscr{C}^\infty_\nu(G), *)$ of $\mathscr{C}^\infty(G)$ such that $*$ is a (differential) star product.*

Remark IV.4 Techniques similar to those indicated here can be applied to other q-algebras (more general quantum groups such as those in [Fr97] and more recent examples, Yangians, etc.). In particular those used in the case of the Jimbo models should be applicable to q-algebras defined by generators and relations. That direction of research has not yet been developed.

Since from any Drinfel′d quantum group we obtain a star product, and since any FRT quantum group can be seen as a restriction of such a star product, we have showed that the data of a "semi-simple" quantum group is equivalent to the data of a star product on $\mathscr{C}^\infty(G)$ satisfying $\Delta(f*g) = \Delta(f)*\Delta(g)$.

Actually the functorial existence results of Etingof and Kazhdan [EK96] on the quantization of Lie bialgebras (see also [En02]) show that the latter is true also for "non semi-simple" quantum groups.

In our framework, we obtained a result in this direction about preferred deformations:

Theorem IV.4 ([BP96]) *Let G be a simply connected Poisson-Lie group such that its associated Lie bialgebra has a preferred quantization. Then there exists a deformation $(\mathscr{C}^\infty_\nu(G),*)$ of $\mathscr{C}^\infty(G)$ such that $*$ is a (differential) star product.*

This can be applied, for example, when $\mathrm{Lie}(G)$ is the double of some Lie algebra, since there exists a preferred quantization [EK96].

It is important to remark that the proof of this theorem does not use the preceding construction. The main argument is an integrability result concerning formal deformations [LP93].

V TOWARDS EXPLICIT REALIZATIONS, I

V.1 Star exponentials and star representations

Let G be a Lie group (connected and simply connected), acting by symplectomorphisms on a symplectic manifold X (e.g. coadjoint orbits in the dual of the Lie algebra $\mathfrak{g}$ of G). The elements $x, y \in \mathfrak{g}$ will be supposed to be realized by functions u_x, u_y in $\mathscr{C}^\infty(X)$ so that their Lie bracket $[x,y]_\mathfrak{g}$ is realized by $\{u_x, u_y\}$.

We define $(g.u)(\xi) = u(g^{-1}.\xi)$ the induced action of G on $\mathscr{C}^\infty(X)$. A very natural problem is the existence of a star product $*$ on $\mathscr{C}^\infty(X)$ such that $g.u*g.v = g.(u*v)$, $\forall u,v \in \mathscr{C}^\infty(X)$, $\forall g \in G$, that is, a G-invariant star product. Explicit examples of such star products will be given later (on some symmetric spaces). But in general, even for a nilpotent G acting on a coadjoint orbit, an invariant star product does not always exists.

This leads to consider a weaker condition: we say that $*$ is *covariant* if there exists a deformation τ of . $(\tau_g(u) = g.u + \nu\ldots)$ such that $*$ is invariant under τ. It can be shown [AC85a] that is equivalent to ask that $\{u_x, u_y\} = [u_x, u_y] \equiv (u*v - v*u)/2\nu$.

Now take a G-covariant star-product $*$, then the map $\mathfrak{g} \ni x \mapsto (2\nu)^{-1}u_x \in \mathscr{C}^\infty(X)$ is a Lie algebra morphism. The appearance of ν^{-1} here and in the trace (see 2.2.1) cannot be avoided and explains why we have often to take into account both ν and ν^{-1}. We can now define the *star exponential*

$$E(e^x) = \mathrm{Exp}(x) \equiv \sum_{n=0}^{\infty} (n!)^{-1}(u_x/2\nu)^{*n} \tag{13}$$

where $x \in \mathfrak{g}$, $e^x \in G$ and the power $*n$ denotes the n^{th} star-power of the corresponding function. By the Campbell–Hausdorff formula one can extend E to a *group homomorphism* $E : G \to (\mathscr{C}^\infty(X)[[\nu,\nu^{-1}]],*)$ where, in the formal series, ν and ν^{-1} are treated as independent parameters for the time being. Alternatively, the values of E can be taken in the algebra $(\mathscr{P}[[\nu^{-1}]],*)$, where $\mathscr{P}$ is the algebra generated by $\mathfrak{g}$ with the $*$-product (it is a representation of the enveloping algebra).
A *star representation* [BFFLS] of G is a distribution $\mathscr{E}$ (valued in $\mathrm{Im}E$) on X defined by

$$D \ni f \mapsto \mathscr{E}(f) = \int_G f(g)E(g^{-1})dg$$

where D is some space of test-functions on G. The corresponding *character* χ is the (scalar-valued) distribution defined by $D \ni f \mapsto \chi(f) = \int_X \mathscr{E}(f)d\mu$, $d\mu$ being a quasi-invariant measure on X.

The character is one of the tools which permit a comparison with usual representation theory. For semi-simple groups it is singular at the origin in irreducible representations, which may require caution in computing the star exponential (13). In the case of the harmonic oscillator that difficulty was masked by the fact that the corresponding representation of $\mathfrak{sl}(2)$ generated by (p^2, q^2, pq) is integrable to a double covering of $\mathrm{SL}(2,\mathbb{R})$ and decomposes into a sum $D(\frac{1}{4}) \oplus D(\frac{3}{4})$: the singularities at the origin cancel each other for the two components.

This theory is now very developed, and parallels in many ways the usual (operatorial) representation theory. A detailed account of all the results would take us too far, but among the most notable one may quote:

(i) An exhaustive treatment of *nilpotent* or *solvable exponential* [AC85b] and even *general solvable* Lie groups [ACL]. The coadjoint orbits are there symplectomorphic to $\mathbb{R}^{2\ell}$ and one can lift the Moyal product to the orbits in a way that is adapted to the Plancherel formula. Polarizations are not required, and "star-polarizations" can always be introduced to compare with usual theory. Wavelets, important in signal analysis, are manifestations of star-products on the (2-dimensional solvable) affine group of $\mathbb{R}$ or on a similar 3-dimensional solvable group [BB98].

(ii) For *semi-simple* Lie groups an array of results exists. Some explicit and autonomous formulas for star exponentials [Fr79] are available. In [ACG, Mr86] a complete treatment of the *holomorphic discrete series* (this includes the case of compact Lie groups) was made, using a kind of Berezin dequantization. Similar techniques have also been used [CGR, Kb98] to find invariant star-products on Kähler and Hermitian symmetric spaces (convergent for an appropriate dense subalgebra). Note however, as shown by recent developments of unitary representations theory (see e.g. [Sc97]), that for semi-simple groups the coadjoint orbits alone are no more sufficient for the unitary dual and one needs far more elaborate constructions.

(iii) For semi-direct products, and in particular for the Poincaré and Euclidean groups, an autonomous theory has also been developed (see e.g. [ACM]).

Comparison with the usual results of "operatorial" theory of Lie group representations can be performed in several ways, in particular by constructing an invariant Weyl transform generalizing (4), finding "star-polarizations" that always exist, in contradistinction with the geometric quantization approach (where at best one can find complex polarizations), study of spectra (of elements in the center of the enveloping algebra and of compact generators) in the sense of (2.2.3.1), comparison of characters, etc. Note also in this context that the pseudodifferential analysis and (non autonomous) connection with quantization developed extensively by Unterberger, first in the case of $\mathbb{R}^{2\ell}$, has been extended to the above invariant context [UU84, UU94]. But our main insistence is that the theory of star representations is an *autonomous* one that can be formulated completely within this framework, based on coadjoint orbits (and some additional ingredients when required).

V.2 Universal Deformation Formulae

For an endomorphism ϕ of an algebra A, write a^ϕ for the right action of ϕ on $a \in A$. It was noticed early in deformation theory (see [Ge68]) that if ϕ and ψ are commuting derivations of A, then the product

$$a * b = ab + \nu a^\phi b^\psi + \frac{\nu^2}{2} a^{\phi^2} b^{\psi^2} + \cdots + \frac{\nu^n}{n!} a^{\phi^n} b^{\psi^n} + \cdots$$

defines an associative multiplication. This deformation can simply be written as

$$a * b = \mu_0((a \otimes b)^{\exp \nu(\phi \otimes \psi)})$$

and can be generalized to any sum of commuting derivations. The Moyal product can be realized in this fashion: take $A = \mathscr{C}^\infty(\mathbb{R}^{2n})$ and $\sum_{i<j} \partial_{x_i} \wedge \partial_{x_j}$ as the infinitesimal.

The first explicit deformation formula involving noncommuting derivations appeared in [CGS]. Suppose that ϕ and ψ are derivations satisfying $[\phi, \psi] = \psi$, and let $\phi^{(n)} = \phi(\phi+1)\ldots(\phi+n-1)$. Then

$$a * b = ab + \nu a^\phi b^\psi + \frac{\nu^2}{2} a^{\phi^{(2)}} b^{\psi^2} + \cdots + \frac{\nu^n}{n!} a^{\phi^{(n)}} b^{\psi^n} + \cdots \tag{14}$$

is associative. As an exponential, this deformation may be written as

$$a * b = \mu_0\big((a \otimes b)^{\exp(\phi \otimes \ln(1+\nu\psi))}\big).$$

As we will see, it is this formula that produces the "Jordanian" quantization of $\mathscr{C}^\infty(SL_2)$.

These are examples of *universal deformation formulae*, a construction of formal deformations inspired by Drinfel′d's twisting concept of introduced in [D89a, D89b]. The basic idea is that one has a deformation solely based upon information about the Lie algebra of derivations, and the formula is independent of the actual algebra in question. Such constructions are examples of a mutual feedback between Hopf algebraic techniques in quantum groups and deformation quantization. The following subsections provide further examples.

Definition V.1 *An element $F \in B \otimes B$ is a twisting element (based on a bialgebra B with comultiplication Δ_B and counit ε_B) if*

1. $(\varepsilon_B \otimes \mathtt{Id})F = 1 \otimes 1 = (\mathtt{Id} \otimes \varepsilon_B)F,$ *and*
2. $F_{12}\left[(\Delta_B \otimes \mathtt{Id})(F)\right] = F_{23}\left[(\mathtt{Id} \otimes \Delta_B)(F)\right].$

The virtue of having such an F is that it can be used to twist the entire category of right B-module algebras and left B-module coalgebras in a uniform way. The following result from [GZ95] is based on the fundamental ideas from [D89b].

Theorem V.1 *Let $F \in B \otimes B$ be a twisting element.*

1. *If A is a right B-module algebra, then $A_F = A(\mu_A \circ F_r, 1_A)$ is an associative algebra.*
2. *If C is a left B-module coalgebra, then $C_F = C(F_l \circ \Delta_C, \varepsilon_C)$ is a coassociative coalgebra.*
3. *If F is invertible and $\Delta'_B = F_l \circ F_r^{-1} \circ \Delta_B$, then $B_F = B(\mu_B, \Delta'_B, 1_B, \varepsilon_B)$ is a k-bialgebra.*
4. *If F is invertible A is a right B-module algebra, then A_F is a right B_F-module algebra.*
5. *If F is invertible and C is a left B-module coalgebra, then C_F is a left B_F-module coalgebra.*

The connection between twisting elements and deformations is the following:

Definition V.2 *A universal deformation formula (UDF) based on a bialgebra B is a twisting element F based on $B[[\nu]]$ of the form*

$$F = 1 \otimes 1 + \nu F_1 + \nu^2 F_2 + \cdots + \nu^n F_n + \cdots$$

where each $F_i \in B \otimes B$.

If F is a UDF then it is clear that A_F, C_F, and B_F as constructed in the theorems above are all deformations. It is clear that the twisting elements defined after Theorem (IV.1) are UDFs with $B = \mathscr{U}\mathfrak{g}[[\nu]]$.

Example V.1

1. *The most basic UDF is based on the bialgebra $B = \mathscr{U}\mathfrak{a}$ where $\mathfrak{a}$ is an abelian Lie algebra. The UDF is the classic exponential formula $F = \exp \nu b$ where $b \in \mathfrak{a} \otimes a$ is arbitrarily chosen. An example of a deformation arising from this UDF arise is the Moyal deformation of $\mathscr{C}^\infty(\mathbb{R}^{2n})$. Another example is the deformation of the standard quantum group $\mathscr{C}^\infty_q(SL(n))$ to the multiparameter family first introduced in [Re90].*

2. *Let $\mathfrak{s}$ is the two-dimensional solvable Lie algebra with basis $\{\phi, \psi\}$ and relation $[\phi, \psi] = \psi$, and set $B = \mathscr{U}\mathfrak{s}$. Then $F = \exp(\phi \otimes \ln(1 + \nu\psi))$ is a UDF.*

3. *Other specific examples of UDF's, including ones not based on enveloping algebras, can be found in [BBMa], [CGW], [CM03], [KLM99], [LS02] and [GZ95].*

Note that, according to Theorems (IV.1) and (V.1), a UDF based on $\mathscr{U}\mathfrak{g}$ produces a preferred deformation $\mathscr{U}\mathfrak{g}$ and matching "covariant" deformations of $\mathscr{C}^\infty(M)$ for every manifold which admits an action of the Lie group G. We shall discuss this topic in more detail later on.

V.3 Crossed products

The literature on Hopf algebras and quantum groups contains a large collection of what we can call generically semi-direct products, or crossed products. These constructions make crucial use of the comultiplication and we will use the standard Sweedler notation for the coproducts [Sw69]: in a coalgebra (H, Δ), $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ and, by coassociativity, $(\mathtt{Id} \otimes \Delta)\Delta(x) = (\Delta \otimes \mathtt{Id})\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$.

Considering crossed products gives explicit, concise and workable formulae. The properties of these structures, as well as for semi-direct products of groups or Lie algebras, may be deduced from the ones of the two original structures. So we can hope to obtain interesting results on, for example, cohomology or representation theories.

We will investigate two original examples: the topological quantum double and the deformation quantizations of some symmetric spaces.

But first, let us give a general idea of what it is. The simplest example of these crossed products is usually called the smash product (see [Sw68, Mn77]):

Definition V.3 *Let B be a bialgebra and C a B-module algebra. The smash product $C\sharp B$ is the algebra constructed on the vector space $C \otimes B$ where the multiplication is defined by*

$$(f \otimes a) \overset{\rightharpoonup}{*} (g \otimes b) = \sum_{(a)} f\ (a_{(1)} \rightharpoonup g) \otimes a_{(2)} b \tag{15}$$

for $f, g \in C$ and $a, b \in B$.

Remark V.1

1. (a) Let H and K be groups and let $\tau : K \to \mathtt{Aut}(H)$ be an action of K on H. This induces a $\mathbb{C}K$-module algebra structure on $\mathbb{C}H$. Then $\mathbb{C}H\sharp\mathbb{C}K \cong \mathbb{C}(H \rtimes K)$, $H \rtimes K$ denoting the semi-direct product of H by K.

   (b) Similarly, for Lie algebras $\mathfrak{h}$ and $\mathfrak{k}$, a Lie algebra homomorphism $\sigma : \mathfrak{k} \to \mathtt{Der}(\mathfrak{h})$ induces a $\mathscr{U}\mathfrak{k}$-module algebra structure on $\mathscr{U}\mathfrak{h}$. Then $\mathscr{U}\mathfrak{h}\sharp\mathscr{U}\mathfrak{k} \cong \mathscr{U}(\mathfrak{h} \rtimes \mathfrak{k})$.

2. The smash product can be seen as the algebraic version of what is called "crossed product" in the C^*-algebra literature [DVDZ, Pe79]. Note that this is an important structure in this context, extensively used, for example, in the works around the Baum-Connes conjecture.

Now let us describe some generalizations:

- This product can be seen in the cohomological interpretation of Sweedler [Sw68] as a representative of the trivial class of a theory of extensions. The formula of the smash product can be "twisted" a little more by some 2-cocycle from $B\otimes B$ to C and is called a crossed product.
- If B and C are bialgebras, C a B-module algebra and B a C-module algebra, with some compatibilities between the two actions, one can write some kind of more "symmetric" formula. S. Majid has called double crossproduct the resulting algebra [Mj90]. This definition leads to a good description of the structure of quantum double introduced by Drinfel′d in [Dr87] (see below for details).
- If C is a bialgebra and B is cocommutative, the natural tensor coproduct on $C\otimes B$ yields a bialgebra structure on $C\sharp B$. If everything is Hopf, $C\sharp B$ can be made Hopf as well [Mn77].
- By dualizing Definition V.3, one gets a coalgebra called the cosmash product. Combining smash and cosmash in order to form a bialgebra leads to the notion of bicrossproduct [Mj90].

Before introducing a new and useful generalization of this kind of definitions let us go back to an application of the double crossproduct to the notion of quantum double in the context of topological Hopf algebras.

V.3.1 Topological quantum double

In [Dr87] Drinfel′d defined the quantum double of $\mathscr{U}_\nu\mathfrak{g}$ (see also [Sts]). This can be adapted to the context of topological Hopf algebras [Bo94].

For this subsection A will denote $\mathscr{D}(G), \mathscr{A}(G), \mathscr{D}_\nu(G)$ or $\mathscr{A}_\nu(G)$.

Definitions If A is described by (A,μ,Δ,S) then $A^*=(A^*,{}^t\Delta,{}^t\mu,{}^tS)$. Define $A^0=A^{*\ co-op}=(A^*,{}^t\Delta,{}^t\mu^{op},{}^tS^{op})$, where $\mu^{op}(x\otimes y):=\mu(y\otimes x)$ and S^{op} is the antipode compatible with μ^{op} and Δ.

If we consider the vector space $A^*\otimes A$, Drinfel′d [Dr87] defines the quantum double as follows:
i) $D(A)\simeq A^0\otimes A$ as coalgebras,
ii) $(f\otimes Id_A).(Id_{A^0}\otimes b)=f\otimes b$,
iii) $(Id_{A^0}\otimes e_s).(e^t\otimes Id_A)=\Delta_s^{kjn}\ \mu^t_{plk}\ S'^p_n\ (e^l\otimes Id_A)\ (Id_{A^0}\otimes e_j)$, where $\{e_s\}$ is a basis of A and $\{e^t\}$ the dual basis.

The Drinfel′d double was expressed [Mj90] in a Sweedler form for dually paired Hopf algebras as an example of a theory of 'double smash products'. Adapting that formulation to our topological context we can now define the double as:

Definition V.4 *The double of A, $D(A)$, is the topological Hopf algebra $(A^*\overline{\otimes}A,\mu_D,{}^t\mu^{op}\otimes\Delta,{}^tS^{op}\otimes S)$ with*

$$\begin{aligned}\mu_D((f\otimes a)\otimes(g\otimes b)) &= \sum_{(a)} f < g\,,S^{op}(a_{(3)})\,?\,a_{(1)} > \otimes a_{(2)}\,b\\ &= \sum_{(a)(g)} < g_{(1)},a_{(1)} > < {}^tS^{op}(g_{(3)}),a_{(3)} > \ f g_{(2)}\otimes a_{(2)} b\end{aligned}$$

where $<\ ,\ >$ denotes the pairing A^*/A, "?" stands for a variable in A and $\overline{\otimes}$ is the completed inductive tensor product.

As topological vector spaces we have $D(A)=A^*\overline{\otimes}A$. Thus $D(A)^*=A\hat{\otimes}A^*$ and $D(A)^{**}=D(A)$. So $D(A)$ is "almost self dual" (it is self dual up to a completion) and is reflexive.

Extension theory

- If A is cocommutative then the product μ_D of $D(A)$ is the *smash product* $\overrightarrow{\mu}$ on $A^0\overline{\otimes}A$

$$\overrightarrow{\mu}\left((f\otimes a)\otimes(g\otimes b)\right)=\sum_{(a)} f(a_{(1)}\rightharpoonup g)\otimes a_{(2)}b$$

  where $\rightharpoonup$ denotes the coadjoint action of A on A^0, $<a\rightharpoonup f,b>=\sum_{(a)}<f,S(a_{(1)})ba_{(2)}>$. This product is the "zero class" of an extension theory, defined by Sweedler [Sw68], classified by a space of 2-cohomology $H^2_{sw}(A,A^0)$. The products are of the form, for τ a 2-cocycle,

$$\overrightarrow{\mu_\tau}\left((f\otimes a)\otimes(g\otimes b)\right)=\sum_{(a)(b)} f(a_{(1)}\rightharpoonup g)\tau(a_{(2)}\otimes b_{(2)})\big)\otimes a_{(3)}b_{(2)}.$$

- The coproduct of $D(A)$ is a smash coproduct for the trivial co-action. We can dualize the theory and, putting the two things together, we obtain an extension theory for bialgebras which is classified by a cohomology space $H^2_{bisw}(A^0,A)$.

So we can ask the following question: are there other possible definitions of the double as an extension of A^0 by A?
We get a partial answer:

Proposition V.1 ([Bo94]) *$H^2_{bisw}\big(\mathscr{D}(G),\mathscr{C}^\infty(G)\big)=\{0\}$ so $D(\mathscr{D}(G))$ is the unique extension of $\mathscr{C}^\infty(G)$ by $\mathscr{D}(G)$.*

V.3.2 L-R smash product

In order to shed light on the general definition which follows, we return to the simplest case of deformation quantization: the Moyal product on $\mathbb{R}^2$. We look at $\mathbb{R}^2$ as $T^*\mathbb{R}\equiv\mathbb{R}\times\mathbb{R}^*$ and therefore can write $\mathscr{C}^\infty(\mathbb{R}^2)\simeq\mathscr{C}^\infty(\mathbb{R})\hat{\otimes}\mathscr{C}^\infty(\mathbb{R}^*)$. We consider first two functions of a special kind in this algebra: $u(x)=u(x_1,x_2)=f(x_1)P(x_2)$ and $v(x)=v(x_1,x_2)=g(x_1)Q(x_2)$ where $f,g\in\mathscr{C}^\infty_0(\mathbb{R})$ and P,Q are polynomials in $\mathsf{Pol}(\mathbb{R}^*)\simeq\mathsf{S}\mathbb{R}$. We can then write the usual coproduct on the symmetric algebra $\mathsf{S}\mathbb{R}$ as $\Delta(P)(x_2,y_2)=P(x_2+y_2)(\overset{\text{notation}}{=}\sum_{(P)}P_{(1)}(x_2)P_{(2)}(y_2))$.

We now look at Formula (8) for the Moyal star product on $\mathbb{R}^2$ and perform on it some formal calculations (we do not discuss the convergence of the integrals involved). Up to a constant (depending on ν) we get:

$$\begin{aligned}
(u*v)(x) &= \int_{\mathbb{R}^2\times\mathbb{R}^2} u(x+y)v(x+z)e^{-\frac{i}{\nu}\Lambda^{-1}(y,z)}dydz\\
&= \int_{\mathbb{R}^2\times\mathbb{R}^2} f(x_1+y_1)P(x_2+y_2)g(x_1+z_1)Q(x_2+z_2)e^{-\frac{i}{\nu}(y_1z_2-y_2z_1)}dy_1dy_2dz_1dz_2\\
&= \int_{\mathbb{R}^2} f(x_1+y_1)Q(x_2+z_2)e^{-\frac{i}{\nu}y_1z_2}dy_1dz_2.\int_{\mathbb{R}^2} g(x_1+z_1)P(x_2+y_2)e^{\frac{i}{\nu}y_2z_1}dy_2dz_1\\
&= \sum_{(P)(Q)}(\partial^+_{Q_{(1)}}f)(x_1)Q_{(2)}(x_2).(\partial^-_{P_{(1)}}g)(x_1)P_{(2)}(x_2) \qquad \text{(up to a constant)}
\end{aligned}$$

with $\partial^\pm_{Q_{(1)}}=Q_{(1)}(\mp i\nu\partial_{x_1})$ (the same for P), since $F^\mp_\nu\left(\alpha F^\pm_\nu(h)(\alpha)\right)(x)=\mp i\nu\partial_x h(x)$ for $h\in\mathscr{C}^\infty_0(\mathbb{R})$ with $F^\pm_\nu(h)(\alpha)$ defined as $\int_{\mathbb{R}}h(x)e^{\mp\frac{i}{\nu}x\alpha}dx$. This suggests the following small generalization of the smash product:

Definition V.5 *Let B be a cocommutative bialgebra and C a B-bimodule algebra (i.e. a B-module algebra for both, left and right, B-module structures). The* **L-R-smash product** *$C\natural B$ is the algebra constructed on the vector space $C\otimes B$ where the multiplication is defined by*

$$(f\otimes a)*(g\otimes b)=\sum_{(a)(b)}(f\leftharpoonup b_{(1)})(a_{(1)}\rightharpoonup g)\otimes a_{(2)}b_{(2)} \tag{16}$$

for $f,g\in C$ and $a,b\in B$.

Proposition V.2 *The L-R smash product is associative.*

In the same spirit, one has

Lemma V.1 *If C is a B-bimodule bialgebra, the natural tensor product coalgebra structure on $C\otimes B$ defines a bialgebra structure to $C\natural B$.*

If C and B are Hopf algebras, $C\natural B$ is a Hopf algebra as well, defining the antipode by

$$\begin{aligned} J_*(f\otimes a) &= \sum_{(a)} J_B(a_{(1)})\rightharpoonup J_C(f)\leftharpoonup J_B(a_{(2)})\otimes J_B(a_{(3)}) \\ &= \sum_{(a)}(1_C\otimes J_B(a_{(1)}))*(J_C(f)\otimes 1_B)*(1_C\otimes J_B(a_{(2)})). \end{aligned} \tag{17}$$

Now by a careful computation, one proves

Proposition V.3 *Let B be a cocommutative bialgebra, C a B-bimodule algebra and $(C\natural B,*)$ their L-R-smash product.*
Let S be a linear automorphism of C (as a vector space). We define:

(i) the product $\bullet^S$ on C by

$$f\bullet^S g=S^{-1}(S(f).S(g)); \tag{18}$$

(ii) the left and right B-module structures, $\overset{S}{\rightharpoonup}$ and $\overset{S}{\leftharpoonup}$, by

$$a\overset{S}{\rightharpoonup} f:=S^{-1}(a\rightharpoonup S(f)) \ \text{ and } \ f\overset{S}{\leftharpoonup} a:=S^{-1}(S(f)\leftharpoonup a); \tag{19}$$

(iii) the product, $*^S$, on $C\otimes B$ by

$$(f\otimes a)*^S(g\otimes b)=T^{-1}(T(f\otimes a)*T(g\otimes b)) \tag{20}$$

where $T:=S\otimes \mathtt{Id}$.

Then $(C,\bullet^S)$ is a B-bimodule algebra for $\overset{S}{\rightharpoonup}$ and $\overset{S}{\leftharpoonup}$ and $*^S$ is the L-R-smash product defined by these structures.
Moreover, if $(C,.,\Delta_C,J_C,\rightharpoonup,\leftharpoonup)$ is a Hopf algebra and a B-bimodule bialgebra, then

$$C_S:=(C,\bullet^S,\Delta_C^S:=(S^{-1}\otimes S^{-1})\circ\Delta_C\circ S,J_C^S:=S^{-1}\circ J_C\circ S,\overset{S}{\rightharpoonup},\overset{S}{\leftharpoonup})$$

is also a Hopf algebra and a B-bimodule bialgebra. Therefore, by Lemma V.1,

$$(C_S\natural B,*^S,\Delta^S=(23)\circ(\Delta_C^S\otimes\Delta_B),J_*^S),$$

is a Hopf algebra for Δ^S the natural tensor product coalgebra structure on $C_S\natural B$ (with $(23):C\otimes C\otimes B\otimes B\to C\otimes B\otimes C\otimes B,\ c_1\otimes c_2\otimes b_1\otimes b_2\mapsto c_1\otimes b_1\otimes c_2\otimes b_2$) and J_*^S the antipode given on $C_S\natural B$ by Lemma V.1. Also, one has

$$\Delta^S=(T^{-1}\otimes T^{-1})\circ(23)\circ(\Delta_C\otimes\Delta_B)\circ T \ \text{ and } \ J_*^S=T^{-1}\circ J_*\circ T$$

with $T=S\otimes\mathtt{Id}$.

V.3.3 Examples in deformation quantization on $T^*(G)$

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $T^*(G)$ its cotangent bundle. We denote by $\mathscr{U}\mathfrak{g}$, $\mathsf{T}\mathfrak{g}$ and $\mathsf{S}\mathfrak{g}$ respectively the enveloping, tensor and symmetric algebras of $\mathfrak{g}$. Let $\mathsf{Pol}(\mathfrak{g}^*)$ be the algebra of polynomial functions on $\mathfrak{g}^*$. We have the usual identifications:

$$\mathscr{C}^\infty(T^*G) \simeq \mathscr{C}^\infty(G \times \mathfrak{g}^*) \simeq \mathscr{C}^\infty(G)\hat{\otimes}\mathscr{C}^\infty(\mathfrak{g}^*) \supset \mathscr{C}^\infty(G) \otimes \mathsf{Pol}(\mathfrak{g}^*) \simeq \mathscr{C}^\infty(G) \otimes \mathsf{S}\mathfrak{g}.$$

First we deform $\mathsf{S}\mathfrak{g}$ via the "parametrized version",$\mathscr{U}_\nu\mathfrak{g}$, of $\mathscr{U}\mathfrak{g}$ defined by

$$\mathscr{U}_\nu\mathfrak{g} = \frac{\mathsf{T}\mathfrak{g}[[\nu]]}{< XY - YX - \nu[X,Y]; X,Y \in \mathfrak{g} >}.$$

$\mathscr{U}_\nu\mathfrak{g}$ is naturally a Hopf algebra with $\Delta(X) = 1 \otimes X + X \otimes 1$, $\varepsilon(X) = 0$ and $S(X) = -X$ for $X \in \mathfrak{g}$. For $X \in \mathfrak{g}$, we denote by $\widetilde{X}$ (resp. $\overline{X}$) the left- (resp. right-) invariant vector field on G such that $\widetilde{X}_e = \overline{X}_e = X$. We consider the following $\mathbb{K}[[\nu]]$-bilinear actions of $B = \mathscr{U}_\nu\mathfrak{g}$ on $C = \mathscr{C}^\infty(G)[[\nu]]$, for $f \in C$ and $\lambda \in [0,1]$:

(i) $(X \rightharpoonup f)(x) = \nu(\lambda - 1)\ (\widetilde{X}.f)(x)$,

(ii) $(f \leftharpoonup X)(x) = \nu\lambda\ (\overline{X}.f)(x)$.

One then has

Lemma V.2 *C is a B-bimodule algebra w.r.t. the above left and right actions (i) and (ii).*

Definition V.6 *We denote by $*_\lambda$ the star product on $(\mathscr{C}^\infty(G) \otimes \mathsf{Pol}(\mathfrak{g}^*))\,[[\nu]]$ given by the L-R-smash product on $\mathscr{C}^\infty(G)[[\nu]] \otimes \mathscr{U}_\nu\mathfrak{g}$ constructed from the bimodule structure of the preceding lemma.*

Proposition V.4 *For $G = \mathbb{R}^n$, $*_{\frac{1}{2}}$ is the Moyal star product (Weyl ordered), $*_0$ is the standard ordered star product and $*_1$ the anti-standard ordered one. In general $*_\lambda$ yields the λ-ordered quantization, within the notation of M. Pflaum [Pf99].*

Remark V.2 In the general case, it would be interesting to compare our λ-ordered L-R smash product with classical constructions of star products on $T^*(G)$ with Gutt's product as one example [Gu83].

V.3.4 Hopf structures

We have discussed (see Lemma V.1) the possibility of having a Hopf structure on $C\natural B$. Let us consider the particular case of $\mathscr{C}^\infty(\mathbb{R}^n)[[\nu]]\natural\mathscr{U}_\nu\mathbb{R}^n = \mathscr{C}^\infty(\mathbb{R}^n)[[\nu]]\natural\mathsf{S}\mathbb{R}^n$ ($\mathbb{R}^n$ is commutative). $\mathsf{S}\mathbb{R}^n$ is endowed with its natural Hopf structure but we also need a Hopf structure on $\mathscr{C}^\infty(\mathbb{R}^n)[[\nu]] = \mathscr{C}^\infty(\mathbb{R}^n) \otimes \mathbb{R}[[\nu]]$. We will not use the usual one. Our alternative structure is defined as follows.

Definition V.7 *We endow $\mathbb{R}[[\nu]]$ with the usual product, the co-product $\Delta(P)(t_1,t_2) := P(t_1+t_2)$, the co-unit $\varepsilon(P) = P(0)$ and the antipode $J(\nu) = -\nu$. We consider the Hopf algebra $(\mathscr{C}^\infty(\mathbb{R}^n), \cdot, \mathbf{1}, \Delta_C, \varepsilon_C, J_C)$, with pointwise multiplication, the unit $\mathbf{1}$ (the constant function of value 1), the coproduct $\Delta_C(f)(x,y) = f(x+y)$, the co-unit $\varepsilon(f) = f(0)$ and the antipode $J_C(f)(x) = f(-x)$. The tensor product of these two Hopf algebras then yields a Hopf algebra denoted by*

$$(\mathscr{C}^\infty(\mathbb{R}^n)[[\nu]], ., \mathbf{1}, \Delta_\nu, \varepsilon_\nu, J_\nu).$$

Note that Δ_ν and J_ν are not linear in ν. We then define, on the L-R smash $\mathscr{C}^\infty(\mathbb{R}^n)[[\nu]]\natural\mathsf{S}\mathbb{R}^n$,

$$\Delta_* := (23) \circ (\Delta_\nu \otimes \Delta_B), \quad \varepsilon_* := \varepsilon_\nu \otimes \varepsilon_B \quad \textit{and} \quad J_* \textit{ as in Lemma V.1 }.$$

Proposition V.5 *$(\mathscr{C}^\infty(\mathbb{R}^n)[[\nu]]\natural\mathsf{S}\mathbb{R}^n, *_\lambda, \mathbf{1} \otimes 1, \Delta_*, \varepsilon_*, J_*)$ is a Hopf algebra.*

Remark V.3 The case $\lambda = \frac{1}{2}$ yields the usual Hopf structure on the enveloping algebra of the Heisenberg Lie algebra.

V.4 UDFs revisited and quantization of symmetric spaces

V.4.1 Introduction

The previous discussion of universal deformation formulae was purely formal. However, the ideas can be extended to include geometric considerations, which allows for more flexibility in the applications.

Let G be a group acting on a set M. Denote by $\tau : G \times M \to M : (g,x) \mapsto \tau_g(x)$ the (left) action and by $\alpha : G \times \mathrm{Fun}(M) \to \mathrm{Fun}(M)$ the corresponding action on the space of (complex valued) functions (or formal series) on M ($\alpha_g := \tau^*_{g^{-1}}$). Assume that on a subspace $\mathbb{A} \subset \mathrm{Fun}(G)$, one has an associative $\mathbb{C}$-algebra product $*^G_{\mathbb{A}} : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ such that

(i) $\mathbb{A}$ is invariant under the (left) regular action of G on $\mathrm{Fun}(G)$,

(ii) the product $*^G_{\mathbb{A}}$ is left-invariant as well i.e. for all $g \in G; a,b \in \mathbb{A}$, one has

$$(L^*_g a) *^G_{\mathbb{A}} (L^*_g b) = L^*_g (a *^G_{\mathbb{A}} b). \tag{21}$$

(In Hopf algebra language, this means that $(\mathbb{A}, *^G_{\mathbb{A}})$ is a $\mathbb{C}G$-module algebra.)

Given a function on M, $u \in \mathrm{Fun}(M)$, and a point $x \in M$, one denotes by $\alpha^x(u) \in \mathrm{Fun}(G)$ the function on G defined as

$$\alpha^x(u)(g) := \alpha_g(u)(x). \tag{22}$$

Then one readily observes that the subspace $\mathbb{B} \subset \mathrm{Fun}(M)$ defined as

$$\mathbb{B} := \{u \in \mathrm{Fun}(M) \,|\, \forall x \in M : \alpha^x(u) \in \mathbb{A}\} \tag{23}$$

becomes an associative $\mathbb{C}$-algebra when endowed with the product $*^M_{\mathbb{B}}$ given by

$$u *^M_{\mathbb{B}} v(x) := (\alpha^x(u) *^G_{\mathbb{A}} \alpha^x(v))(e) \tag{24}$$

(e denotes the neutral element of G). Of course, all this can be defined for right actions as well.

Definition V.8 *A pair $(\mathbb{A}, *^G_{\mathbb{A}})$ is called a (left)* universal deformation *of G, while Formula (24) is called the associated* universal deformation formula *(briefly* UDF*).*

In the present article, we will be concerned with the case where G is a Lie group. The function space $\mathbb{A}$ will be either
- a functional subspace (or a topological completion) of $\mathscr{C}^\infty(G,\mathbb{C})$ containing the smooth compactly supported functions in which case we will talk about *strict deformation* (following Rieffel [Ri89]),
or,
- the space $\mathbb{A} = \mathscr{C}^\infty(G)[[\nu]]$ of formal power series with coefficients in the smooth functions on G in which case, we'll speak about *formal deformation*. In any case, we'll assume the product $*^G_{\mathbb{A}}$ admits an asymptotic expansion of star-product type:

$$a *^G_{\mathbb{A}} b \sim ab + \frac{\nu}{2i}\mathbf{w}(\mathrm{d}u, \mathrm{d}v) + o(\nu^2) \qquad (a,b \in \mathscr{C}^\infty_c(G)),$$

where $\mathbf{w}$ denotes some (left-invariant) Poisson bivector on G[BFFLS]. In the strict cases considered here, the product will be defined by an integral three-point kernel $K \in \mathscr{C}^\infty(G \times G \times G)$:

$$a *^G_{\mathbb{A}} b(g) := \int_{G\times G} a(g_1)\, b(g_2) K(g, g_1, g_2) \mathrm{d}g_1\, \mathrm{d}g_2 \qquad (a,b \in \mathbb{A})$$

where $\mathrm{d}g$ denotes a normalized left-invariant Haar measure on G. Moreover, our kernels will be of *WKB type* [We94, Ks94] i.e.:

$$K = A e^{\frac{i}{\nu}\Phi},$$

with A (the *amplitude*) and Φ (the *phase*) in $\mathscr{C}^\infty(G\times G\times G,\mathbb{R})$ being invariant under the (diagonal) action by left-translations.
Note that in the case where the group G acts smoothly on a smooth manifold M by diffeomorphisms: $\tau : G\times M \to M : (g,x) \mapsto \tau_g(x)$, the first-order expansion term of $u *^M_{\mathbb{B}} v, \quad u,v \in C^\infty(M)$ defines a Poisson structure $\mathbf{w}^M$ on M which can be expressed in terms of a basis $\{X_i\}$ of the Lie algebra $\mathfrak{g}$ of G as:

$$\mathbf{w}^M = [\mathbf{w}_e]^{ij} X_i^* \wedge X_j^*, \tag{25}$$

where X^* denotes the fundamental vector field on M associated to $X \in \mathfrak{g}$.

V.4.2 Elementary solvable symplectic symmetric spaces and their strict quantization

Symmetric spaces

Definition V.9 ([Bi95]) *A* symplectic symmetric space *is a triple* (M,ω,s)*, where* (M,ω) *is a smooth connected symplectic manifold and* $s : M\times M \to M$ *is a smooth map such that:*

(i) for all x in M, the partial map $s_x : M \to M : y \mapsto s_x(y) := s(x,y)$ *is an involutive symplectic diffeomorphism of* (M,ω) *called the* symmetry *at x.*

(ii) For all x in M, x is an isolated fixed point of s_x*.*

(iii) For all x and y in M, one has $s_x s_y s_x = s_{s_x(y)}$*.*

Two symplectic symmetric spaces (M,ω,s) *and* (M',ω',s') *are* isomorphic *if there exists a symplectic diffeomorphism* $\varphi : (M,\omega) \to (M',\omega')$ *such that* $\varphi s_x = s'_{\varphi(x)}\varphi$*.*

We denote by G the *transvection group* of (M,s) (i.e. the subgroup of $\mathtt{Aut}(M,\omega,s)$ generated by $\{s_x \circ s_y\,;\, x,y \in M\}$)

Definition V.10 *Let* $(\mathfrak{g},\sigma)$ *be an* involutive algebra*, that is,* $\mathfrak{g}$ *is a finite dimensional real Lie algebra and* σ *is an involutive automorphism of* $\mathfrak{g}$*. Let* Ω *be a skewsymmetric bilinear form on* $\mathfrak{g}$*. Then the triple* $(\mathfrak{g},\sigma,\Omega)$ *is called a* symplectic triple *if the following properties are satisfied:*

1. *Let* $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ *where* $\mathfrak{k}$ *(resp.* $\mathfrak{p}$*) is the* $+1$ *(resp.* -1*) eigenspace of* σ*. Then* $[\mathfrak{p},\mathfrak{p}] = \mathfrak{k}$ *and the representation of* $\mathfrak{k}$ *on* $\mathfrak{p}$*, given by the adjoint action, is faithful.*
2. Ω *is a Chevalley 2-cocycle for the trivial representation of* $\mathfrak{g}$ *on* $\mathbb{R}$ *such that* $\forall X \in \mathfrak{k}$, $i(X)\Omega = 0$*. Moreover, the restriction of* Ω *to* $\mathfrak{p}\times\mathfrak{p}$ *is nondegenerate.*

The dimension of $\mathfrak{p}$ *defines the* dimension *of the triple. Two such triples* $(\mathfrak{g}_i,\sigma_i,\Omega_i)$ $(i = 1,2)$ *are* isomorphic *if there exists a Lie algebra isomorphism* $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ *such that* $\psi\circ\sigma_1 = \sigma_2\circ\psi$ *and* $\psi^*\Omega_2 = \Omega_1$*.*

Proposition V.6 ([Bi95]) *There is a bijective correspondence between the isomorphism classes of simply connected symplectic symmetric spaces* (M,ω,s) *and the isomorphism classes of symmetric triples* $(\mathfrak{g},\sigma,\Omega)$*.*

Proposition V.7 *To each symplectic symmetric space (corresponding to a class* $[(\mathfrak{g},\sigma,\Omega)]$*), we can associate an other triple* $\tau = (\mathfrak{h}(\mathfrak{g}),\sigma,\Omega)$ *such that*

- $\mathfrak{h}(\mathfrak{g})$ *is a one dimensional central extension of* $\mathfrak{g}$*.*
- $(\mathfrak{h}(\mathfrak{g}),\sigma)$ *is an involutive Lie algebra such that if* $\mathfrak{h}(\mathfrak{g}) = \mathfrak{l}\oplus\mathfrak{p}$ *is the decomposition w.r.t.* σ *one has* $[\mathfrak{p},\mathfrak{p}] = \mathfrak{l}$*,*
- Ω *is a Chevalley 2-coboundary (i.e.* $\Omega = \delta\xi\,, \quad \xi \in \mathfrak{h}(\mathfrak{g})^*$*) such that* $i(\mathfrak{l})\Omega = 0$ *and* $\Omega|_{\mathfrak{p}\times\mathfrak{p}}$ *is symplectic.*

Such a triple is called exact triple.

Elementary solvable symplectic symmetric spaces In Definition V.11 below, we define a particular type of solvable symmetric spaces which we call elementary. It has been proven ([Bi98], Proposition 3.2) that every solvable symmetric space is realized through a sequence of split extensions by Abelian (flat) factors successively taken over an elementary solvable symmetric space. We therefore consider elementary solvable symmetric spaces as the "first induction step" when studying solvable symmetric spaces.

Definition V.11 *A symplectic symmetric space (M,ω,s) is called an* elementary solvable *symplectic symmetric space if its associated exact triple $(\mathfrak{h}(\mathfrak{g}),\sigma,\Omega=\delta\xi)$ (see Lemma V.7) is of the following type.*

(i) The Lie algebra $\mathfrak{h}(\mathfrak{g})$ is a split extension of Abelian Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$:

$$0\to\mathfrak{b}\longrightarrow\mathfrak{h}(\mathfrak{g})\longrightarrow\mathfrak{a}\to 0.$$

(ii) The automorphism σ preserves the splitting $\mathfrak{h}(\mathfrak{g})=\mathfrak{b}\oplus\mathfrak{a}$.

Such an exact triple (associated to an elementary solvable symplectic symmetric space) is called an elementary solvable exact triple.

Observe that, since $\mathfrak{a}\cap\mathfrak{k}\subset\mathfrak{a}\cap[\mathfrak{h}(\mathfrak{g}),\mathfrak{h}(\mathfrak{g})]=0$, one has $\mathfrak{a}\subset\mathfrak{p}$. Therefore $\mathfrak{b}=\mathfrak{k}\oplus\mathfrak{l}$, with $\mathfrak{l}\subset\mathfrak{p}$. Moreover, since $\mathfrak{l}$ and $\mathfrak{a}$ are Abelian and Ω is nondegenerate, the subspaces $\mathfrak{a}$ and $\mathfrak{l}$ of $\mathfrak{p}$ are dual Lagrangians.

Now let (M,ω,s) be an elementary solvable symplectic symmetric space with associated exact triple $(\mathfrak{h}(\mathfrak{g}),\sigma,\Omega=\delta\xi)$ as above. In a neighborhood U of the origin, the map

$$\mathfrak{p}=\mathfrak{a}\times\mathfrak{l}\to M:(a,l)\mapsto\exp(a)\exp(l).o \tag{26}$$

turns out to be a Darboux chart when $U\subset\mathfrak{p}$ has the symplectic structure $\Omega=\delta\xi$. Moreover, there exists a unique immersion $\phi:U\cap\mathfrak{a}\to\mathfrak{a}$ such that in the local coordinate system (26), one has the following linearization property:

$$\xi(\sinh(a)l)=\xi[\phi(a),l]; \tag{27}$$

where, for $a\in\mathfrak{a}$ we set $\sinh(a):=\frac{1}{2}(\exp(\rho(a))-\exp(-\rho(a)))\in\mathtt{End}(\mathfrak{b})$. This immersion is called the *twisting map*.

Proposition V.8 *An elementary solvable symplectic symmetric space is strictly geodesically convex if and only if its associated twisting map extends to $\mathfrak{a}$ as a global diffeomorphism of $\mathfrak{a}$. In this case, the Darboux chart (26) extends as a global symplectomorphism $(\mathfrak{p},\Omega)\to(M,\omega)$.*

Quantization Associated to the twisting map one has a three-point function $S\in\mathscr{C}^\infty(M\times M\times M,\mathbb{R})$ called the *WKB-phase* of the elementary solvable symplectic symmetric space:

$$S(x_0,x_1,x_2):=\xi\left(\oint_{0,1,2}\sinh(a_0-a_1)l_2\right); \tag{28}$$

where $\oint_{0,1,2}$ stands for cyclic summation and where $x_i=(a_i,l_i)\quad(i=0,1,2)$. The phase S turns out to be invariant under the (diagonal) action of the symmetries $\{s_x\}_{x\in M}$ on $M\times M\times M$. This will be the essential constituent of the associative oscillatory kernel defining a symmetry-invariant strict quantization on every elementary solvable symplectic symmetric space. We now recall this construction as in [Bi00].

Definition V.12 *For a compactly supported function $u\in C^\infty_c(\mathfrak{p})$, identifying $\mathfrak{l}^*$ with $\mathfrak{a}$, we denote by $\tilde{u}\in\mathscr{C}^\infty(\mathfrak{a}\times\mathfrak{a})$ its partial Fourier transform:*

$$\tilde{u}(a,\alpha):=\int_{\mathfrak{l}}e^{i\Omega(\alpha,l)}u(a,l)\mathrm{d}l. \tag{29}$$

We also denote by $\phi_\nu : \mathfrak{a} \to \mathfrak{a}$ the one-parameter family of twisting maps:

$$\phi_\nu(a) := \frac{2}{\nu}\phi(\frac{\nu}{2}a). \tag{30}$$

For $u, v \in C^\infty_c(\mathfrak{p})$, we set

$$< u \mid v >_\nu := \int_{\mathfrak{a}\times\mathfrak{a}} \tilde{u}(a,\alpha)\overline{\tilde{v}(a,\alpha)}\,|Jac_{\phi^{-1}}(\alpha)|\,\mathrm{d}a\,\mathrm{d}\alpha. \tag{31}$$

The pair $(\mathscr{C}^\infty(\mathfrak{p}), <,>_\nu)$ is a pre-Hilbert space, and we denote by $\mathscr{H}_\nu$ its Hilbert completion.

The Hilbert product $<,>_\nu$ turns out to be symmetry-invariant on $\mathscr{C}^\infty_c(M)$. The action of the transvection group then extends by continuity to an isometric action on $\mathscr{H}_\nu$.

Theorem V.2 *[Bi00] Let (M,ω,s) be a strictly geodesically convex elementary solvable symplectic symmetric space. Realize it symplectically as $(\mathfrak{p} = \mathfrak{a}\times\mathfrak{l},\Omega)$, and define the two-point function $A \in \mathscr{C}^\infty(M\times M)$ by:*

$$A(x_1,x_2) := |Jac_\phi(a_1 - a_2.)| \tag{32}$$

This function is called the WKB-amplitude *and turns out to be symmetry-invariant. In this notation, one has the following.*

(i) For all $\nu \in \mathbb{R}\backslash\{0\}$ and $u,v \in \mathscr{C}^\infty_c(M)$, the formula:

$$u *_\nu v(x_0) := \int_{M\times M} u(x_1)\,v(x_2)\,A(x_1,x_2)\,e^{\frac{i}{\nu}S(x_0,x_1,x_2)}\,\mathrm{d}x_1\,\mathrm{d}x_2 \tag{33}$$

*extends as an associative product on $\mathscr{H}_\nu$ (*dx *denotes some normalization of the symplectic volume on (M,ω)). Moreover, (for suitable u,v and x_0) the stationary phase method yields a power series expansion of the form*

$$u *_\nu v(x_0) \sim uv(x_0) + \frac{\nu}{2i}\{u,v\}(x_0) + o(\nu^2); \tag{34}$$

where $\{\,,\}$ denotes the symplectic Poisson bracket on (M,ω).

(ii) The pair $(\mathscr{H}_\nu, *_\nu)$ is a topological Hilbert algebra which the transvection group of (M,ω,s) acts on by automorphisms.

A classical procedure then produces a similar result in the C^*-context, see [Bi00] for details.

Remark V.4 Whether a symmetric space is strictly geodesically convex is of course entirely encoded in the spectral content of the splitting endomorphism $\rho : \mathfrak{a} \to \mathtt{End}(\mathfrak{b})$. This is discussed in detail in [Bi00].

Before applying these results, let us discuss heuristically the ideas that led to this quantization. Denote by $\tilde{G}$ the group naturally obtained from the Lie algebra $\mathfrak{h}(\mathfrak{g})$. Proposition V.8 gives a global Darboux chart from $\mathfrak{p} \simeq \mathfrak{a}\times\mathfrak{l} \simeq M$ to the coadjoint orbit $\mathscr{O} = Ad^*(\tilde{G}).\xi \subset \mathfrak{h}(\mathfrak{g})^*$. Denote by $\lambda_X \in \mathscr{C}^\infty(\mathfrak{p})$ the Hamiltonian function associated to the infinitesimal action of $X \in \mathfrak{h}(\mathfrak{g})$ and by $*^M_\nu$ the standard Moyal star product on $(\mathfrak{p},\Omega)$. So, using ideas coming from star representation theory (see Section V.1) we remark that $[\lambda_X,\lambda_Y]_{*^M_\nu} = \frac{\nu}{i}\{\lambda_X,\lambda_Y\}$, that is, $*^M_\nu$ is $\mathfrak{h}(\mathfrak{g})$-covariant. This covariance allows us to define a representation of $\mathfrak{h}(\mathfrak{g})$ on the space $\mathscr{C}^\infty(\mathscr{O})[[\nu]]$, $\rho_\nu(X)u = \frac{i}{\nu}[\lambda_X,u]_{*^M_\nu}$.

Through the partial Fourier transform F (29) ($F(u) := \tilde{u}$) we obtain a representation on $\mathscr{C}^\infty(\mathfrak{a}\times\mathfrak{a})[[\nu]]$, $\tilde{\rho}_\nu$ s.t. $\tilde{\rho}_\nu(X)\tilde{u} = X_{\mathfrak{a}}.\tilde{u} + c_\nu(X)\tilde{u}$. The expression of the cocycle c_ν is very similar to the one of the "twisting map" ϕ_ν (30). So now let us consider a deformation of the partial Fourier transform defined by $Z_\nu(u)(a,\alpha) = \tilde{u}(a,\phi_\nu(\alpha))$. Defining the commutative product $\bullet_\nu$ on $\mathscr{C}^\infty(\mathfrak{a}\times\mathfrak{a})[[\nu]]$ by $f \bullet_\nu g = Z_\nu(Z_\nu^{-1}(f).Z_\nu^{-1}(g))$, calculations show that $\bullet_\nu$ is invariant under $\tilde{\rho}_\nu$.

More, we have $Z_\nu^{-1} \circ \tilde{\rho}_\nu(X) \circ Z_\nu = X^*$, X^* being the vector field induced on M by the action of $\mathfrak{h}(\mathfrak{g})$. That means that the action of $\mathfrak{h}(\mathfrak{g})$ on the "underlying manifold" M_ν of $(\mathscr{C}^\infty(\mathfrak{a}\times\mathfrak{a})[[\nu]], \bullet_\nu)$ is equivalent to the one of $\mathfrak{h}(\mathfrak{g})$ on M. So, "going back" to M by F^{-1} we define $T_\nu = F^{-1}\circ Z_\nu$ and we get a formal product on $\mathscr{C}^\infty(\mathfrak{p})[[\nu]]$ defined by

$$u *_\nu v = T_\nu^{-1}(T_\nu u *_\nu^M T_\nu v) \tag{35}$$

which is invariant under the coadjoint action of $\tilde{G}$ on $\mathscr{O} = \mathfrak{p}$.

V.4.3 UDF and Hopf algebra structure

In [BBMa] we define a specific class of Lie groups called elementary solvable pre-symplectic Lie groups. Let g be a group in this class. By the quantification of elementary solvable symmetric spaces described above, we get a left G-invariant star product on a G-invariant (under the regular action of G) algebra of functions on G, $\mathbb{A}$. That is, we get a UDF $(\mathbb{A}, *_{\mathbb{A}}^G)$ (see definition V.8) for every group G in this class. The formulas are convergent (strict deformation quantization). By asymptotic expansion we obtain also a formal UDF on $\mathscr{C}^\infty(G)[[\nu]]$.

Now, at the formal level, we will obtain compatible coproducts and antipode, seeing the above quantization as a L-R smash product (definition V.5).

To do that let us recall that the obtained star products are of the form

$$u *_\nu v = T_\nu^{-1}(T_\nu u *_\nu^M T_\nu v)$$

with $T = F^{-1}\circ(\mathtt{Id}\otimes\phi_\nu^*)\circ F$. But F does not act on the $\mathfrak{a}$ variable, so we can see it as a map from $\mathscr{C}^\infty(\mathfrak{l})$ to $\mathscr{C}^\infty(\mathfrak{a})$. So we have $T = \mathtt{Id}\otimes S$ with $S = F^{-1}\circ\phi_\nu^*\circ F$. Considering that

$$\mathscr{C}^\infty(M) \simeq \mathscr{C}^\infty(\mathfrak{p}) \simeq \mathscr{C}^\infty(\mathfrak{a})\hat{\otimes}\mathscr{C}^\infty(\mathfrak{l}) \underset{a\simeq\mathfrak{l}^*}{\simeq} \mathscr{C}^\infty(\mathfrak{l}^*)\hat{\otimes}\mathscr{C}^\infty(\mathfrak{l}) \supset Pol(\mathfrak{l}^*)\otimes\mathscr{C}^\infty(\mathfrak{l}) \underset{\mathfrak{l}\ \text{Abelian}}{\simeq} \mathscr{U}\mathfrak{l}\otimes\mathscr{C}^\infty(\mathfrak{l}),$$

it is easy to show

Proposition V.9 *The formal version of the invariant WKB-quantization of an elementary solvable symplectic symmetric spaces defined in Theorem V.2 is a L-R-smash product of the form $*^S$ (cf. Proposition V.3).*

Corollary V.1 *The UDF's for elementary solvable pre-symplectic Lie groups admit compatible coproducts and antipodes.*

Then [BBMb] gives a UDF and an associated Hopf algebra structure for every three dimensional solvable Lie group. The dressing action of the "book" group on $SU(2)$ is particularly investigated.

VI TOWARDS EXPLICIT REALIZATIONS, II

In the next sections, we revisit in more detail the ideas that were hinted upon in Section (IV.1). Recall that any formal deformation of $\mathscr{U}\mathfrak{g}$ or $\mathscr{C}^\infty(G)$ is preferred and all are produced by modified twisting elements (solutions of (11) with non-trivial Φ) or twisting elements (solutions of (11) with $\Phi = 1\otimes1\otimes1$). In both cases, $F = 1\otimes1+\nu r+O(\nu^2)$. We will be particularly interested in the infinitesimal r.

The deformation of $\mathscr{U}\mathfrak{g}$ or $\mathscr{C}^\infty(G)$ is called "triangular" if F is a twisting element. A standard cohomological equivalence argument can be invoked to ensure that the infinitesimal r lies in $\mathfrak{g}\wedge\mathfrak{g}$ and satisfies the *classical Yang-Baxter equation* (CYBE)

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \tag{36}$$

We will frequently call this type of infinitesimal a *triangular r-matrix.*

The deformation of $\mathscr{U}\mathfrak{g}$ or $\mathscr{C}^\infty(G)$ is called *quasi-triangular* if F is a modified twisting element. A deep result of Drinfel′d [D89b] asserts that, in the quasi-triangular case, the element Φ is uniquely determined up to a certain natural equivalence. The element Φ is known as the *associator* and it has deep connections to the theory of quasi-Hopf algebras, the Kniznik-Zamolodchikov equations, and a theorem of Kohno regarding certain representations of braid groups. It also enters in a crucial way in the operadic demonstration [Ta98, Hi03] of the Kontsevich formality theorem [Ko97] (which gives the existence of a star-product on every Poisson manifold).

We do not have the time to discuss these matters in this survey. In this case however, the infinitesimal r still satisfies the CYBE but $r \in \mathfrak{g}\otimes\mathfrak{g}$ is no longer skew. Instead, we have $r + r_{21} = \Omega$ where Ω is the Casimir element in $\mathfrak{g}\otimes\mathfrak{g}$ chosen with respect to some fixed invariant bilinear form on $\mathfrak{g}$. In this case, the infinitesimal r is called a *quasi-triangular* r-matrix, and it necessarily follows that $\tilde{r} = r - \dfrac{\Omega}{2} \in \mathfrak{g}\wedge\mathfrak{g}$ will satisfy the *modified classical Yang-Baxter equation* (MCYBE)

$$[\tilde{r}_{12},\tilde{r}_{13}] + [\tilde{r}_{12},\tilde{r}_{23}] + [\tilde{r}_{13},\tilde{r}_{23}] \in (\mathfrak{g}\wedge\mathfrak{g}\wedge\mathfrak{g})^{\mathfrak{g}}. \tag{37}$$

Given an r-matrix of either type, it is natural to seek the twisting element F (guaranteed to exist by [Dr85, D89b]) whose infinitesimal is r. In the explicit sense, the results are quite scarce. Indeed, as previously mentioned, F is unknown for all quasi-triangular r-matrices including, in particular, the "standard" solution which serves as the infinitesimal for the quantum groups $\mathscr{U}_q(\mathfrak{g})$ and $\mathscr{C}_q^\infty(G)$.

VI.1 Preferred $*$-products for the standard quantum n-space

Since the F is unknown, it should be no surprise that the preferred $*$-products have not been exhibited even for $\mathscr{C}_q^\infty(SL(n))$. However, we do have the preferred $*$-products for the covariant quantization, $\mathscr{C}_q^\infty(\mathbb{C}^n)$, of the function algebra $\mathscr{C}^\infty(\mathbb{C}^n)$. If x_i are the coordinate functions on $\mathbb{C}^n$, then $\mathscr{C}_q^\infty(\mathbb{C}^n)$ is characterized by the relations $x_ix_j = qx_jx_i$ for $i < j$. In the classical case, the action of the group SL_n on $\mathbb{C}^n$ makes $\mathscr{C}^\infty(\mathbb{C}^n)$ into an $\mathscr{C}^\infty(SL(n))$-comodule algebra. This means that $\mathscr{C}^\infty(\mathbb{C}^n)$ is an $\mathscr{C}^\infty(SL(n))$-comodule with the compatibility condition that the coaction map is an algebra homomorphism. The same is true at the quantized level: $\mathscr{C}_q^\infty(\mathbb{C}^n)$ is an $\mathscr{C}_q^\infty(SL(n))$-comodule algebra. A preferred covariant deformation in this case is a $*$-product on $\mathscr{C}^\infty(\mathbb{C}^n)$ which is compatible with both the original unchanged (on all elements) comodule structure map (on all elements) and the preferred quantization of $\mathscr{C}^\infty(SL(n))$ to $\mathscr{C}_q^\infty(SL(n))$.

In order to describe the preferred $*$-products for the quantum linear space, we need some combinatorial notation. Define the q-factorial to be

$$m!_q = \frac{(1-q^m)(1-q^{m-1})\cdots(1-q)}{(1-q)^m}.$$

and for $\lambda \in \mathbb{N}^n$ with $|\lambda| = \lambda_1 + \cdots + \lambda_n$ let

$$\langle\lambda\rangle_q = \frac{|\lambda|!_q}{\lambda_1!_q\cdots\lambda_n!_q}$$

be the q-multinomial coefficient. Without the subscript, $\langle\lambda\rangle$ will denote the usual multinomial coefficient. The symbol X^λ will denote the monomial $x_1^{\lambda_1}\cdots x_n^{\lambda_n}$. Note that by linearity, it is only necessary to consider the product of monomials. The ordinary commutative multiplication is simply expressible as $X^\lambda X^\nu = X^{\lambda+\nu}$. Finally, for $\lambda,\nu \in \mathbb{N}^n$ set

$$(\lambda:\nu) = \sum_{i=1}^n\sum_{j>i}\lambda_i\nu_j.$$

Theorem VI.1 *[GGS90, G92] The preferred deformation for the quantum linear space $\mathscr{C}_q(\mathbb{C}^n)$ is given by*

$$X^\lambda * X^\nu = \left(\frac{\langle\lambda\rangle_{q^2}\langle\nu\rangle_{q^2}\langle\lambda+\nu\rangle}{\langle\lambda\rangle\langle\nu\rangle\langle\lambda+\nu\rangle_{q^2}} \right)^{1/2} q^{(\lambda:\nu)}\, X^{\lambda+\nu}. \tag{38}$$

The presence of the factor $q^{(\lambda:\nu)}$ is clear as it comes directly from the defining relations $x_i x_j = q x_j x_i$ of $\mathscr{C}_q^\infty(\mathbb{C}^n)$. The other numerical factor in the formula may be viewed as a ratio of norms and its necessity is less obvious. Note that some of the products in (38) are undefined when q is a root of unity which helps explain from the deformation quantization viewpoint that the standard quantum groups at roots of unity have different behavior than at generic specializations of q. Upon seeing this formula in 1990, Ludwig Faddeev was startled and claimed that, within a year, we would have the preferred $*$-products on all of $\mathscr{C}_q^\infty(SL(n))$. Unfortunately, time has not been kind to his prediction as this problem is unfortunately still open.

Formula (38) of Theorem VI.1 was recently rediscovered by Blohmann in [Bl03] using q-Clebsch-Gordan coefficients.

VI.2 Triangular r-matrices and twists

Suppose that r is a triangular r-matrix (a solution of (VI). Drinfel′d has given a procedure in [Dr85] which, given a triangular r, produces the twisting F. It is generally difficult though to extract the exact form of F since the construction uses, among other things, the Campbell-Baker-Hausdorff formula for the series of $\ln(e^x e^y)$ where $x, y \in \mathfrak{g}$. In the "strong" explicit sense, the twisting F is known only for certain classes of r-matrices including the "Jordanian" solution for $\mathfrak{sl}(2)$ and several of its generalizations.

Let us recall the basic classification scheme (see [BD82]) for triangular r-matrices. Our formulation follows that of Stolin [St91]. A Lie algebra $\mathfrak{f}$ is quasi-Frobenius if there exists a non-degenerate two-cocycle $\phi : \mathfrak{f}\wedge\mathfrak{f} \to \mathbb{C}$. The connection to triangular r-matrices is that the cocycle condition for the bilinear form ϕ is equivalent to ϕ^{-1} (the element of $\mathfrak{f}\wedge\mathfrak{f}$ whose matrix of coefficients is the inverse of the matrix of ϕ) satisfying the classical Yang-Baxter equation. The problem of finding triangular r-matrices up to equivalence is then reduced to listing the quasi-Frobenius Lie algebras, their normalizers, and calculating the second cohomology group $H^2(\mathfrak{f}, \mathbb{C})$. A simple Lie algebra $\mathfrak{g}$ is never quasi-Frobenius but if $r \in \mathfrak{g}\wedge\mathfrak{g}$ is classical r-matrix, then there is a unique quasi-Frobenius "carrier" subalgebra $\mathfrak{f} \subset \mathfrak{g}$ for which $r \in \mathfrak{f}\wedge\mathfrak{f}$. Note that such an $\mathfrak{f}$ is necessarily even-dimensional. This approach does not give a constructive classification of triangular r-matrices as there is no effective way to find all quasi-Frobenius Lie subalgebras of $\mathfrak{g}$.

The easiest example of the foregoing is when $\mathfrak{f}$ is an abelian subalgebra of $\mathfrak{g}$. Then any $r \in \mathfrak{f}\wedge\mathfrak{f}$ is a classical r-matrix and it is elementary to check that the corresponding twisting element is $F = \exp(\nu r)$. We have already seen that this twisting element (=UDF) produces, in particular, the Moyal deformation.

Perhaps the most well known example of a triangular r-matrix is based on the Borel subalgebra of $\mathfrak{sl}(2)$ which is generated by h and e with $[h, e] = 2e$. The r-matrix $h \wedge e$ is the infinitesimal of the "Jordanian" quantization of $\mathscr{C}^\infty(SL(2))$, which is its unique (up to equivalence) triangular quantization. The twisting element can be derived using (14) and comes out to be $F = \exp(\frac{h}{2} \otimes \ln(1+\nu e))$. Note that the infinitesimal is $h \otimes e$ and not $h \wedge e$. These are equivalent r-matrices and the twisting element with infinitesimal $h \wedge e$ is given in [GZ95].

There are various extensions of the Jordanian r-matrix to $\mathfrak{sl}(n)$ with $n > 2$. In particular, the triangular r-matrix

$$r = (e_{11} - e_{nn}) \wedge e_{1n} + 2\sum_{j=2}^{n-1} e_{1j} \wedge e_{jn}$$

was exhibited in [GGS90]. The element F for this family has been called an *extended Jordanian* twist and was derived independently in [GZ95] and [KLM99], although the latter reference has the following more compact form of the twisting element:

$$F = \exp\left\{2\nu \sum_{i=2}^{n-1} e_{1i} \otimes e_{in} e^{-\sigma}\right\} \exp\{(e_{11} - e_{nn}) \otimes \sigma\}$$

where $\sigma = \frac{1}{2}\ln(1+2\nu e_{1n})$. Note that the carriers for this family are contained in the Borel subalgebra of $\mathfrak{sl}(n)$, and are hence solvable.

Another interesting generalization of the Jordanian twist is a family of r-matrices whose carriers, in contrast with the above examples, are non-solvable subalgebras of $\mathfrak{sl}(n)$. In particular, the carrier algebra is the (Frobenius) Lie algebra $\mathfrak{p}_1$ which denotes the maximal parabolic subalgebra generated by all simple positive root vectors $e_{i,i+1}$, the Cartan subalgebra of traceless matrices, and the simple negative root vectors $e_{i+1,i}$ *except* e_{21}. For example, when $n = 3$, this Lie algebra consists of all traceless matrices of the form

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

The r-matrix with carrier $\mathfrak{p}_1 \subset \mathfrak{sl}(n)$ was discovered in [GG97]. For $n = 3$ the explicit form is

$$r = (2e_{11} - e_{22} - e_{33}) \wedge e_{12} + (e_{11} + e_{22} - 2e_{33}) \wedge e_{23} + e_{13} \wedge e_{32}.$$

The twist for this family is only known in the case of $n = 3$; it has the form of a product of an extended Jordanian twist (as illustrated above) with a “deformed” Jordan type twist. Details of this interesting twist can be found in [LS02].

VI.3 Quasi-triangular r-matrices: The Belavin-Drinfel′d classification

We now turn to the case of quasi-triangular r-matrices. First, some good news. There is a complete constructive classification of all such $r \in \mathfrak{g} \otimes \mathfrak{g}$ when $\mathfrak{g}$ is simple. This is the famous Belavin-Drinfel′d classification, see [BD84]. To describe this classification we need some notation. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. The root system will be denoted by Δ and Γ will be a set of simple roots. Let $\Omega_0 \in \mathfrak{g} \otimes \mathfrak{g}$ be the restriction of the Casimir element Ω to $\mathfrak{h} \otimes \mathfrak{h}$.

Definition VI.1 *A Belavin-Drinfel′d triple for $\mathfrak{g}$ is a triple (Γ_1, Γ_2, T) where Γ_i are subsets of the positive simple roots of $\mathfrak{g}$ and $T : \Gamma_1 \to \Gamma_2$ is a bijection which preserves the invariant bilinear form and for all $\alpha \in \Gamma_1$, there exists $k > 1$ such that $T^k \notin \Gamma_1$.*

Given such a triple (henceforth simply denoted T), an element $s \in \mathfrak{h} \otimes \mathfrak{h}$ is called T-admissible if

$$s + s_{21} = \Omega_0 \qquad \text{and} \qquad (T(\alpha) \otimes 1)s + (1 \otimes \alpha)s = \Omega_0$$

for all $\alpha \in \Gamma_1$. A T-admissible s is always of the form $s = \tilde{s} + \Omega_0/2$ where $\tilde{s} \in \mathfrak{h} \wedge \mathfrak{h}$. The set of all admissible $\tilde{s}$ forms a linear subvariety of $\mathfrak{h} \wedge \mathfrak{h}$ whose dimension is $\binom{d}{2}$ where $d = \#(\Gamma - \Gamma_1)$.

The map T can be extended to an isomorphism of Lie subalgebras $T : \mathfrak{g}_1 \to \mathfrak{g}_2$ where $\mathfrak{g}_i$ is the Lie subalgebra of $\mathfrak{g}$ generated by the simple roots in Γ_i. Choose $e_\alpha \in \mathfrak{g}_\alpha$ such that $(e_\alpha, e_{-\alpha}) = 1$ and $T(e_\alpha) = e_{T\alpha}$ and define an ordering on Δ by $\alpha \prec \beta$ if $T^k\alpha = \beta$ for some positive integer k. View $\mathfrak{g} \wedge \mathfrak{g}$ as a subset of $\mathfrak{g} \otimes \mathfrak{g}$ according to the identification $x \wedge y = 1/2(x \otimes y - y \otimes x)$. The spectacular result of Belavin and Drinfel′d is the content of the next theorem.

Theorem VI.2 *[BD84] Let $\mathfrak{g}$ be a simple complex Lie algebra and suppose that T is a triple. Then for every admissible s, the element*

$$r = s + \sum_{\alpha > 0} e_{-\alpha} \otimes e_\alpha + 2 \sum_{\substack{\alpha,\beta > 0 \\ \alpha \prec \beta}} e_{-\alpha} \wedge e_\beta$$

is a quasi-triangular solution to the Yang-Baxter equation satisfying $r_{12}+r_{21}=\Omega$. Moreover, every such solution is of this form up to the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}\otimes\mathfrak{g}$.

The standard solution, $r_{st}=\Omega_0+\sum_{\alpha>0}e_{-\alpha}\otimes e_\alpha$, corresponds to the empty triple (with $\Gamma_i=\emptyset$) and $\tilde{s}=0$. It is the infinitesimal for the quantum groups $\mathscr{U}_q(\mathfrak{g})$ and $\mathscr{C}_q^\infty(G)$. For this triple, any $s=\Omega_0+\tilde{s}$ with $\tilde{s}\in\mathfrak{h}\wedge\mathfrak{h}$ is admissible so the dimension of this family of solutions is $\binom{l}{2}$ where l is the rank of $\mathfrak{g}$.

At the other extreme, there are certain triples for which there is a unique T-admissible element. For $\mathfrak{g}=\mathfrak{sl}_n$, these are the *generalized Cremmer-Gervais* triples, see [GG97]. If $\{\alpha_1,\ldots,\alpha_{n-1}\}$ are the simple positive roots and i is relatively prime to n, then the triple T_i with

$$\Gamma_1=\Gamma-\{\alpha_{n-i}\},\qquad \Gamma_2=\Gamma-\{\alpha_m\},\qquad T(\alpha_j)=\alpha_{i+j \mod n} \tag{39}$$

is a generalized Belavin-Triple. The r-matrix where $m=1$ serves as the infinitesimal of the Cremmer-Gervais R-matrix, see [CG90]. The original approach of Cremmer-Gervais made no mention of the Belavin-Drinfel'd triple; the connection was first made in [GGS93].

VI.4 Boundary solutions of the classical Yang-Baxter equation

Suppose $r\in\mathfrak{g}\wedge\mathfrak{g}$ is a solution of the modified classical Yang-Baxter equation (VI) $[r_{12},r_{13}]+[r_{12},r_{23}]+[r_{13},r_{23}]=c\omega$ where ω is a $\mathfrak{g}$-invariant element in $\mathfrak{g}\wedge\mathfrak{g}\wedge\mathfrak{g}$ and c is a scalar. It should be intuitively clear that as c tends to zero, r tends to a solution of the classical Yang-Baxter equation. This theory of this type of "degeneration" was made precise in [GG97], and it is essentially the concept of contraction which was elaborated upon earlier in this paper.

Let $\mathscr{M}$ be the set of all solutions to the modified classical Yang-Baxter equation in the projective space $\mathbb{P}(\mathfrak{g}\wedge\mathfrak{g})$ of lines in $\mathfrak{g}\wedge\mathfrak{g}$. Similarly, $\mathscr{C}$ will denote the solutions to the classical Yang-Baxter equation in $\mathbb{P}(\mathfrak{g}\wedge\mathfrak{g})$.

Theorem VI.3 *[GG97]*

1. *The set $\mathscr{M}$ is a quasi-projective variety (i.e. an open subset of $\overline{\mathscr{M}}$, its closure).*

2. *Any element in the closure of $\mathscr{M}$ not lying in $\mathscr{M}$ is contained in $\mathscr{C}$.*

3. *There exist elements of $\mathscr{C}$ which are not in $\overline{\mathscr{M}}$.*

In light of the theorem, call an r which lies in $\overline{\mathscr{M}}$ but not $\mathscr{M}$ a *boundary solution* of the classical Yang-Baxter equation.

Question VI.1 *Is there a reasonable constructive classification of the boundary r-matrices? As stated earlier, no such classification seems likely for all triangular r-matrices. But the relation of the boundary solutions to the constructively known set $\mathscr{M}$ gives plausibility to a positive answer to this question.*

An easy way to construct boundary r matrices is the following: Take an ad-nilpotent element $x\in\mathfrak{g}$ and any modified classical r-matrix r. Then $\exp(\xi\,\mathrm{ad}_x)\cdot r$ is necessarily of the form $r+\xi r_1+\cdots+\xi^m r_m$. Dividing the result by ξ^m and letting ξ approach zero then shows that r_m must be a boundary solution of the classical Yang-Baxter equation. The extended Jordanian and parabolic triangular r-matrices discussed earlier were first realized with this construction. The extended Jordanian r lies in the boundary of an orbit of the standard solution to the modified classical Yang-Baxter equation, see [GGS90]. The more interesting example is the parabolic r-matrix. It lies in the closure of an orbit of the Cremmer-Gervais solution to the modified classical Yang-Baxter equation. What is striking here is that both the parabolic solution and the Cremmer-Gervais solution are uniquely determined by the first root of $\mathfrak{sl}(n)$. The parabolic subalgebra is the unique r-matrix with carrier $\mathfrak{p}_1$ (determined by deleting the first negative root), and the Cremmer-Gervais triple is the unique one whose Γ_2 omits only the first root. There should be a parallel situation with the modified Cremmer-Gervais triples, each of which is uniquely determined by an integer relatively prime to n. Elashvili has proved in [E82] that the parabolic subalgebra $\mathfrak{p}_i$ of $\mathfrak{sl}(n)$ determined by deleting $\alpha_i = e_{i+1,i}$ is quasi-Frobenius if and only if i and n are relatively prime.

Conjecture VI.1 *Suppose i is relatively prime to n. Then the triangular r-matrix whose carrier is the maximal parabolic subalgebra $\mathfrak{p}_i$ lies in the closure of the orbit of the modified Cremmer-Gervais quasi-triangular r-matrix determined by the triple T_i (defined in formula (VI.3)).*

In unpublished work, the second and third authors have verified this conjecture by direct computation in the case $n = 5$ and $i = 2$.

VI.5 The GGS "Magic" Formula

The central aspect of the FRT approach is a quantum Yang-Baxter matrix $R \in \mathtt{End}(V \otimes V)$ where V is the vector representation of G. The matrix R is used to provide commutation relations for the generators which define the quantized Hopf algebra of functions. Given a solution r to the classical Yang-Baxter equation (triangular or quasi-triangular), a natural question is quantize it to an R-matrix of the form $R = 1 + \nu r + \nu^2 r_2 + \cdots$ which satisfies the quantum Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ (QYBE).

If r is triangular and F is its corresponding twisting element, then $\mathscr{R} = FF_{21}^{-1}(\mathscr{U}\mathfrak{g} \otimes \mathscr{U}\mathfrak{g})[[\nu]]$ is a universal solution to the QYBE. Specializing to the vector representation then gives a quantum Yang-Baxter matrix. However, it is hard to be explicit here because the F is unknown in the strong explicit sense, except in some cases previously mentioned. However, if r is quasi-triangular, there is the Belavin-Drinfel′d classification and an interesting question is to quantize these r-matrices. For general simple $\mathfrak{g}$ there is no satisfactory answer as of yet, but for $\mathfrak{sl}(n)$ there is a remarkable explicit quantization called the *GGS Formula*, see [GGS93]. Our esteemed colleague Yvette Kosmann-Schwarzbach calls this the *magic* formula because of its simplicity and the remarkable fact that such a simple formula can work for all quasi-triangular r-matrices.

For the standard solution r_{st} the corresponding R-matrix

$$R_{st} = q\sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i > j} e_{ij} \otimes e_{ji} \tag{40}$$

is well-known. In addition to the quantum Yang-Baxter equation, PR_{st} satisfies the Hecke relation $(PR - q)(PR + q^{-1}) = 0$ where P is the matrix representing the interchange of factors in $V \otimes V$. If $\tilde{s} \in \mathfrak{h} \wedge \mathfrak{h}$ then recall that $r_{st} + \tilde{s}$ is a quasi-triangular r-matrix associated to the empty triple. Its quantization has the elementary form $q^{\tilde{s}} R_{st} q^{\tilde{s}}$ and was first given in [Re90], a paper which initiated a flurry of activity in the area of multiparameter quantum groups.

Until 1993, the only other specific R-matrix was that given by Cremmer and Gervais in [CG90]. Then a conjecture was made by Gerstenhaber, Giaquinto and Schack which gave a formula which was conjectured to produce an R-matrix from each Belavin-Drinfel′d r-matrix for $\mathfrak{sl}(n)$. This became known as the "GGS Conjecture." Evidence of the conjecture's validity was confirmed by computer programs for all triples up through $\mathfrak{sl}(5)$ in [GH98], and for all triples up through $\mathfrak{sl}(12)$ in [Sch99]. Up to a natural notion of equivalence, there are 210,300 triples for $\mathfrak{sl}(12)$.

To describe the GGS formula, we need some notation. Let $r_{st} + s + a$ be a Belavin-Drinfel'd quasi-triangular r-matrix. Recall that $s = \tilde{s} + \Omega_0/2$ with $\tilde{s} \in \mathfrak{h} \otimes \mathfrak{h}$. Set

$$\varepsilon = ar_{st} + r_{st}a + a^2.$$

For $M \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ we use the Kronecker notation $M = \sum M_{ik}^{jl} e_{ij} \otimes e_{kl}$. Define

$$\tilde{a} = \sum a_{ij}^{kl} q^{a_{ij}^{kl} \varepsilon_{ij}^{kl}} e_{ij} \otimes e_{kl}$$

Theorem VI.4 *[GGS93, Sch00] Let $r_{st}+s+a$ be a Belavin-Drinfel'd quasi-triangular r-matrix, and define*

$$R_{GGS} = q^{\tilde{s}}(R_{st} + (q-q^{-1})\tilde{a})q^{\tilde{s}}. \tag{41}$$

Then R_{GGS} satisfies the quantum Yang-Baxter equation which quantizes r, and PR_{GGS} satisfies the Hecke relation.

The proof of the conjecture is due to Schedler [Sch00] and is rather complicated. In related work, Etingof, Schedler, and Schiffmann gave an (explicit) quantization of the so-called dynamical r-matrices for simple Lie algebras [ESS]. A special case of their construction gives a universal quantization of any quasi-triangular r-matrix. Even though the construction is "explicit", it is a formidable task to perform computations with the universal quantization in a specific representation. Indeed, Schedler begins his proof of the GGS conjecture with the ESS universal quantization and then he proceeds to evaluate it in the tensor square of the vector representation. After 25 pages of lengthy combinatorial computations, he finds out that the result is exactly the GGS matrix! Something is wanting, however, for a more elementary proof as the statement of the GGS conjecture is just an assertion about specific linear transformation of a finite dimensional vector space.

Question VI.2

1. *Find a similar GGS-type formula which quantizes the quasi-triangular r-matrices for the other classical series (B,C,D) of simple Lie algebras. So far, no progress has been made in this direction.*
2. *Find a "boundary GGS formula." By this, we mean formula which quantizes the boundary r-matrices. For example, it was proved in [GG98a] that for any triangular r-matrix in $\mathfrak{sl}(\mathfrak{n}) \wedge \mathfrak{sl}(n)$ which lies in the boundary of the standard solution $\sum_{i>j} e_{ij} \wedge e_{ji}$ to the MCYBE, then $\exp(\xi r)$ is a boundary solution to the QYBE.*

In a matter related to the second question, Hodges and Endelman have shown in [EH00] that the Cremmer-Gervais R-matrix degenerates to a QYBE matrix which quantizes the parabolic triangular r-matrix associated to the parabolic subalgebra $\mathfrak{p}_1$ of $\mathfrak{sl}(n)$. This may be seen as the quantum version of the degeneration of the Cremmer-Gervais quasi-triangular r-matrix to the triangular parabolic r-matrix.

VI.6 Quantizing in representation space

Let V be the vector representation of a simple complex Lie algebra $\mathfrak{g}$. In the last section, we exhibited a canonical QYBE matrix $R \in \mathtt{End}(V \otimes V)$ associated to each Belavin-Drinfel′d infinitesimal for $\mathfrak{sl}(n)$. The FRT formalism then may be used to produce commutation relations for the generators of a quantization $\mathbb{C}_R[SL(n)]$ of the coordinate Hopf algebra of $SL(n)$. Each R-matrix also enjoys many other nice properties. For example, the operator $\hat{R} = PR$ has eigenvalues q and $-q^{-1}$ with multiplicities $\frac{n^2+n}{2}$ and $\frac{n^2-1}{2}$, respectively. These numbers coincide with the multiplicities of the eigenspaces corresponding to the eigenvalues ± 1 of the permutation operator P.

What the R-matrix does not do, however, is *quantize* the module $V \otimes V$ – that is transform the $+1$-eigenspace for P to the q-eigenspaces for $\hat{R}$ and the -1-eigenspace for P to the $-q^{-1}$-eigenspace for $\hat{R}$. Some explanation is in order to explain more precisely what we mean here. In the classical ($q=1$) case the module $V \otimes V$ splits into a direct sum $V^+ \oplus V^-$, where V^+ consists of the symmetric vectors (eigenvalue 1 for P) and V^- is the space of skew-symmetric vectors (eigenvalue -1 for P). In the quantized case the same idea holds. Specifically, $V \otimes V \cong V_q^+ \oplus V_q^-$ where V_q^+ is the space of q-symmetric vectors (eigenvalue q for $\widehat{R}$) and V_q^- is the space of q-skew-symmetric vectors (eigenvalue $-q^{-1}$) for $\widehat{R}$. What we seek is a *quantizing transformation* $Q \in \mathtt{End}(V \otimes V)$ which takes V^+ to V_q^+ and V^- to V_q^-. If the Belavin-Drinfel′d infinitesimal of R is r, then it is known that any possible quantizing transformation Q has infinitesimal $\tilde{r} = r - \frac{\Omega}{2}$, the solution to the MCYBE associated to r. Ideally we would like to have a canonical (explicit of course!) simple formula for the quantizing transformation Q as a function of $\tilde{r}$.

For the standard solution R_{st}, we do have a pleasant description of Q which we now describe. The solution to the MCYBE associated to r_{st} is $\tilde{r}_{st} = \sum_{i>j} e_{ij} \wedge e_{ji}$. Let $e_1, \ldots, e_n$ be the standard basis of $V \cong \mathbb{C}^n$. It will be convenient to use the inner product on V defined by $(e_i, e_j) = \delta_{ij}$. This extends to one on $V \otimes V$ in which the set of all $e_i \otimes e_j$ forms an orthonormal basis. Then an elementary calculation shows that

$$V_q^+ = \left\{ \{e_i \otimes e_i \mid 1 \le i \le n\} \quad \bigcup \quad \left\{ \frac{q e_i \otimes e_j + e_j \otimes e_i}{\sqrt{1+q^2}} \mid i < j \right\} \right\}$$

and

$$V_q^- = \left\{ \left\{ \frac{e_i \otimes e_j - q e_j \otimes e_i}{\sqrt{1+q^2}} \mid i < j \right\} \right\}.$$

Note that these are orthogonal bases and that setting $q = 1$ gives orthogonal bases for the eigenspaces $V^\pm$ of P.

Theorem VI.5 *[GGS90] Let $Q = \exp(\nu \tilde{r}_{st})$ and $q = \sec\nu - \tan\nu$. Then $Q(e_i \otimes e_i) = e_i \otimes e_i$ for each i and for all $i < j$*

$$Q\left[\frac{e_i \otimes e_j + e_j \otimes e_i}{\sqrt{2}}\right] = \frac{q e_i \otimes e_j + e_j \otimes e_i}{\sqrt{1+q^2}}, \quad \textit{and} \quad Q\left[\frac{e_i \otimes e_j - e_j \otimes e_i}{\sqrt{2}}\right] = \frac{e_i \otimes e_j - q e_j \otimes e_i}{\sqrt{1+q^2}}$$

and

$$\hat{R}_{st} = \frac{1}{\cos\nu}(Q^{-1} P Q - \sin\nu).$$

Thus Q is an orthogonal quantizing transformation and $\hat{R}_{st}$ (and hence also R_{st}) is easily recoverable from Q. In fact, the transformation Q performs the quantization eigenvector by eigenvector, which is quite desirable. In the simplest case of $n = 2$, Q is a rotation in the plane spanned by $e_1 \otimes e_2$ and $e_2 \otimes e_1$. The striking aspect of this theorem is that Q, being just an exponential, is produced in a very elementary way solely based on infinitesimal data. Thus we recover R_{st} in a particularly simple way. Also note that the choice of $q = \sec\nu - \tan\nu$ is absolutely necessary for the theorem. From the viewpoint of deformation quantization, it turns out that this seems to be the natural choice for the parameter q, and not e^ν, or $e^{i\nu}$ as many (including us!) authors tend to use.

It was shown in [GGS90] (see also [Bl03]) that if $F \in \mathscr{U}(\mathfrak{sl}(n)) \otimes \mathscr{U}(\mathfrak{sl}(n))[[\nu]]$ is the modified twisting element which gives the preferred deformation of $\mathscr{C}^\infty(SL(n))$ to $\mathscr{C}_q^\infty(SL(n))$, then $(\rho \otimes \rho)F = Q^{-1}$ where $\rho : \mathfrak{sl}(n) \to \mathtt{End}(V)$ is the vector representation. Thus, the inverse of the quantizing transformation Q coincides with the image of the modified twisting element in the vector representation.

Optimally, we would like to have such an elementary quantizing transformation as a simple function of the corresponding $\tilde{r}_{st}$ for the orthogonal and symplectic Lie algebras. Recent work of second and third authors shows that this is indeed possible, but the construction is more subtle. The difficulty lies in the fact that for these Lie algebras, $V \otimes V$ contains a one-dimensional $\mathfrak{g}$-module, a vector which represents the defining bilinear form of $\mathfrak{g}$. For the orthogonal Lie algebras, this vector splits off of the symmetric vectors and in the symplectic case it splits off of the skew vectors. A similar decomposition occurs in the quantized cases. This difficulty is parallelled in fact that the operator $\hat{R}_{st}$ has three eigenvalues. For type B_n, they are q,$-q^{-1}$ and q^{1-N} where $N = 2n+1$, and the remaining types of C_n and D_n have a similarly formed third eigenvalue. Unfortunately, space prohibits a detailed description of the quantizing transformation in these cases. We can just say what it is not: unlike the A_n series, the exponential of $\tilde{r}_{st}$ does not produce the quantizing transformation. Nevertheless, like many deformation problems, the exponential function plays a substantial role in the construction of the quantizing transformation.

References

[Ab80] Abe E., *Hopf algebras*. Cambridge Tracts in Mathematics, **74**. Cambridge University Press, 1980.

[AL03] Alekseev A. and Lachowska A. "Invariant $*$-products on coadjoint orbits and the Shapovalov pairing," `math.QA/0308100`.

[ACG] Arnal D., Cahen M. and Gutt S. "Representations of compact Lie groups and quantization by deformation," *Bull. Acad. Royale Belg.* **74** (1988), 123–141; "Star exponential and holomorphic discrete series," *Bull. Soc. Math. Belg.* **41** (1989), 207–227.

[AC79] Arnal D. and Cortet J-C. "Geometrical theory of contractions and representations," *J. Math. Phys.* **20** (1979), 556–563.

[AC85a] Arnal D. and Cortet J-C., "Nilpotent Fourier transform and applications," *Lett. Math. Phys.* **9** (1985), 25–34.

[AC85b] Arnal D. and Cortet J-C., "Star-products in the method of orbits for nilpotent Lie groups," *J. Geom. Phys.* **2** (1985), 83–116.

[AC90] Arnal D. and Cortet J-C., "Représentations star des groupes exponentiels," *J. Funct. Anal.* **92** (1990), 103–135.

[ACL] Arnal D., Cortet J-C. and Ludwig J., "Moyal product and representations of solvable Lie groups," *J. Funct. Anal.* **133** (1995), 402–424.

[ACM] Arnal D., Cortet J-C. and Molin P., "Star-produit et représentation de masse nulle du groupe de Poincaré," *C.R. Acad. Sci. Paris Sér. A* **293**, 309–312 (1981).

[BFLS] Basart H., Flato M., Lichnerowicz A. and Sternheimer D., "Deformation theory applied to quantization and statistical mechanics," *Lett. Math. Phys.* **8** (1984), 483–494.

[BFFLS] Bayen F., Flato M., Fronsdal C., Lichnerowicz A., and Sternheimer D., "Deformation theory and quantization I. Deformations of symplectic structures," *Ann. Physics* **111** (1978), 61–110. "Deformation theory and quantization II. Physical applications," *Ann. Physics* **111** (1978), 111–151.

[BD82] Belavin A. and Drinfel′d V., "Solutions of the classical Yang-Baxter equation for simple Lie algebras," (Russian), *Funktsional. Anal. i Prilozhen.* **16** (1982), 1–29.

[BD84] Belavin A. and Drinfel′d V., "Triangle equations and simple Lie algebras," *Mathematical Physics Reviews* **4**, 93 – 165, Harwood Academic Publ., Chur 1984.

[Bers] Bers L. *Selected works of Lipman Bers.* Part 1. Papers on complex analysis. Edited by Irwin Kra and Bernard Maskit. American Mathematical Society, Providence, RI, 1998. "Quasiconformal mappings and Teichmüller's theorem," *Analytic functions* pp. 89–119, Princeton Univ. Press, Princeton, N.J. 1960.

[BB98] Bertrand J. and Bertrand P. "Symbolic calculus on the time-frequency half-plane," *J. Math. Phys.* **39** (1998), 4071–4090.

[Bg96] Bidegain F., "A candidate for a noncompact quantum group," *Lett. Math. Phys.*, **36** (1996), 157–167.

[BP95] Bidegain F. and Pinczon G., “A star-product approach to noncompact quantum groups,” *Lett. Math. Phys.* **33** (1995), 231–240 (`hep-th/9409054`).

[BP96] Bidegain F. and Pinczon G., “Quantization of Poisson-Lie groups and applications,” *Comm. Math. Phys.*, **179** (1996), 295–332.

[Bi95] Bieliavsky P,. *Espaces symétriques symplectiques*, Ph.D. thesis, Université Libre de Bruxelles, 1995.

[Bi98] Bieliavsky P., “Four-dimensional simply connected symplectic symmetric spaces,” *Geom. Dedicata* **69** (1998), 291–316.

[Bi00] Bieliavsky P., “Strict quantization of solvable symmetric spaces,” *J. Sympl. Geom.* **1** (2002), 269–320.

[BBMa] Bieliavsky P., Bonneau P. and Maeda Y., “Universal Deformation Formulae, Symplectic Lie groups and Symmetric Spaces,” `math.QA/0308189` (2003).

[BBMb] Bieliavsky P., Bonneau P. and Maeda Y., “Universal Deformation Formulae for Three-Dimensional Solvable Lie groups,” `math.QA/0308188` (2003).

[BM02] Bieliavsky P. and Maeda Y., “Convergent star product algebras on “$ax+b$”,” *Lett. Math. Phys.* **62** (2002), 233–243.

[BM01] Bieliavsky P. and Massar M., “Oscillatory integral formulae for left-invariant star products on a class of Lie groups,” *Lett. Math. Phys.* **58** (2001), 115–128.

[Bl03] Blohmann C., “Covariant realization of quantum spaces as star products by Drinfel′d twists,” *J. Math. Phys.* **44** (2003), 4736–4755.

[Bo92] Bonneau P., “Cohomology and associated deformations for not necessarily co-associative bialgebras,” *Lett. Math. Phys.*, **26** (1992), 277–283.

[Bo94] Bonneau P., “Topological quantum double,” *Rev. Math. Phys.*, **6** (1994), 305–318.

[BFGP] Bonneau p., Flato M., Gerstenhaber M., and Pinczon G., “The hidden group structure of quantum groups: strong duality, rigidity and preferred deformations,” *Comm. Math. Phys.*, **161** (1994), 125–156.

[BFP] Bonneau P., Flato M., and Pinczon G., “A natural and rigid model of quantum groups,” *Lett. Math. Phys.*, **25** (1992), 75–84.

[dB23] de Broglie L., “Ondes et quanta,” *C.R. Acad. SCi. Paris* **177** (1923), 507–510. *Recherches sur la théorie des quanta*, Thèse, Paris (1924). *La mécanique ondulatoire*, Gauthier-Villars, Paris 1928.

[CGR] Cahen M., Gutt S. and Rawnsley J., “Quantization of Kähler manifolds IV,” *Lett. Math. Phys.* **34** (1995), 159–168.

[CGW] Caldararu A., Giaquinto A., and Witherspoon S., “Algebraic deformations arising from orbifolds with discrete torsion,” *Jour. Pure Appl. Alg.*, **187** (2004), 51–70.

[CGS] Coll V., Gerstenhaber M., and Giaquinto A., “An explicit deformation formula with noncommuting derivations,” *Ring theory 1989*, Israel Math. Conf. Proc., **1**, 396–403, Weizmann, Jerusalem 1989.

[CDV] Connes A. and Dubois-Violette M., "Moduli space and structure of noncommutative 3-spheres," *Lett. Math. Phys.* **66** (2003), 99–121 (`math.QA/0308275`).

[CL01] Connes A. and Landi G., "Noncommutative manifolds, the instanton algebra and isospectral deformations," *Comm. Math. Phys.* **221** (2001), 141–159.

[CK99] Connes A. and Kreimer D., "Lessons from Quantum Field Theory — Hopf Algebras and Spacetime Geometries," *Lett. Math. Phys.* **48** (1999), 85–96.

[CM04] Connes A. and Marcoli M., "From Physics to Number Theory via Noncommutative Geometry. Part I: Quantum Statistical Mechanics of Q-lattices," `math.NT/0404128`.

[CM03] Connes A. and Moscovici A., "Modular Hecke Algebras and their Hopf Symmetry," *Moscow Math. J.* **4** (2004), 67–109 (`math.QA/0301089`). "Rankin-Cohen Brackets and the Hopf Algebra of Transverse Geometry," *ibid.* 111–130 (`math.QA/0304316`).

[CG90] Cremmer E. and Gervais J-L., "The quantum group structure associated with nonlinearly extended Virasoro algebras." *Comm. Math. Phys.* **134** (1990), 619–632.

[DFST] Dito G., Flato M., Sternheimer D. and Takhtajan L., "Deformation Quantization and Nambu Mechanics," *Comm. Math. Phys.* **183** (1997) 1–22 (`hep-th/9602016`).

[DS02] Dito G. and Sternheimer D., "Deformation Quantization: Genesis, Developments and Metamorphoses," in: *Deformation quantization* (G.Halbout ed.), IRMA Lectures in Math. Theoret. Phys. **1**, 9–54. Walter de Gruyter, Berlin 2002 (`math.QA/0202168`).

[DN01] Douglas M. and Nekrasov N., "Noncommutative Field Theory," *Rev.Mod.Phys.* **73** (2001), 977–1029 (`hep-th/0106048`).

[DVDZ] Drabant B., Van Daele A. and Zhang Y., "Actions of multiplier Hopf algebras," *Commun. Algebra* **27** (1999), 4117–4172.

[Dr85] Drinfel′d V., "Hopf algebras and the quantum Yang-Baxter equation," *Dokl. Akad. Nauk SSSR* **283** (1985), 1060–1064.

[Dr87] Drinfel′d V., "Quantum groups," in *Proceedings of the International Congress of Mathematicians*, Vol. 1–2 (Berkeley, Calif., 1986), 798–820, Amer. Math. Soc. Providence, RI 1987.

[D89a] Drinfel′d V, "Almost cocommutative Hopf algebras," *Algebra i Analiz*, **1** (1989), 30–46.

[D89b] Drinfel′d V., "Quasi-Hopf algebras," *Algebra i Analiz*, **1** (1989), 114–148.

[D93] Drinfel′d V., "On Poisson homogeneous spaces of Poisson-Lie groups," *Teoret. Mat. Fiz.*, **95** (1993), 226–227; translation in *Theoret. and Math. Phys.*, **95** (1993), 524–525.

[E82] Elashvili, A., "Frobenius Lie algebras," *Funktsional. Anal. i Prilozhen.* **16** (1982), 94–95.

[EH00] Endelman R. and Hodges T., "Generalized Jordanian *R*-matrices of Cremmer-Gervais type," *Lett. Math. Phys.* **52** (2000), 225–237.

[En02] Enriquez B., "A cohomological construction of quantization functors of Lie bialgebras," `math.QA/0212325`. "On quantization functors of Lie bialgebras," The 2000 Twente Conference on Lie Groups (Enschede), *Acta Appl. Math.* **73** (2002), 133–140.

[EK96] Etingof P. and Kazhdan D., "Quantization of Lie bialgebras I," *Selecta Math. (N.S.)*, **2** (1996), 1–41; II, III *ibid.* **4** (1998), 213–231, 233–269; IV, V *ibid.* **6** (2000), 79–104, 105–130; "Quantization of Poisson algebraic groups and Poisson homogeneous spaces," in *Symétries quantiques* (Les Houches, 1995), 935–946, North-Holland, Amsterdam 1998.

[ESS] Etingof P., Schedler T., and Schiffmann O., "Explicit quantization of dynamical r-matrices for finite dimensional semisimple Lie algebras," *J. Amer. Math. Soc.* **13** (2000), 595–609.

[FRT] Faddeev L.D., Reshetikhin N.Yu. and Takhtajan L.A., "Quantization of Lie groups and Lie algebras," in *Algebraic analysis, Vol. I*, pp. 129–139. Academic Press, Boston, MA 1988.

[Fe94] Fedosov B.V., "A simple geometrical construction of deformation quantization," *J. Diff. Geom.* **40** (1994), 213–238. *Deformation quantization and index theory*, Mathematical Topics **9**, Akademie Verlag, Berlin (1996).

[Fl82] Flato M., "Deformation view of physical theories," *Czechoslovak J. Phys.* **B32** (1982), 472–475.

[FLS] Flato M., Lichnerowicz A. and Sternheimer D. "Deformations of Poisson brackets, Dirac brackets and applications," *J. Math. Phys.* **17** (1976), 1754–1762. "Crochets de Moyal-Vey et quantification," *C.R. Acad. Sci. Paris Sér. A* **283** (1976), 19–24.

[FN57] Frölicher, A. and Nijenuis, A., "A theorem on stability of complex structures," *Proc. Nat. Acad. Sci., U.S.A.* **43** (1957), 239–241.

[Fr79] Frønsdal C., "Some ideas on quantization," *Rep. Math. Phys.* **15** (1979), 111–145.
Frønsdal C. and Sternheimer D. "Explicit relations for Deformation Quantization in a Lie algebra context" (work in progress).

[Fr97] Frønsdal C., "Generalization and exact deformations of quantum groups," *Publ. Res. Inst. Math. Sci.* **33** (1997), 91–149.

[Fr02] Frønsdal C., "Harrison cohomology and abelian deformation quantization on algebraic varieties," in : *Deformation quantization* (G.Halbout ed.), IRMA Lect. Math. Theor. Phys. **1**, 149–161, de Gruyter, Berlin 2002 (hep-th/0109001.)

[Ge63] Gerstenhaber M., "The cohomology structure of an associative ring," *Ann. Math.* **78** (1963), 267–288.

[Ge64] Gerstenhaber M., "On the deformation of rings and algebras," *Ann. Math.* **79** (1964), 59–103.

[Ge68] Gerstenhaber M., "On the deformations of rings and algebras, III," *Ann. Math.* **88** (1968), 1–34.

[GG97] Gerstenhaber M. and Giaquinto A., "Boundary solutions of the classical Yang-Baxter equation," *Lett. Math. Phys.* **40** (1997), 337–353.

[GG98a] Gerstenhaber M. and Giaquinto A., "Boundary solutions of the quantum Yang-Baxter equation and solutions in three dimensions," *Lett. Math. Phys.* **44** (1998), 131–141.

[GG98b] Gerstenhaber M. and Giaquinto A. "Compatible deformations," in: E.L. Green and B. Huisgen-Zimmermann (eds.), *Trends in the Representation Theory of Finite Dimensional Algebras*, Contemporary Mathematics **229**, 159–168, American Mathematical Society, Providence 1998.

[GGS90] Gerstenhaber M., Giaquinto A. and Schack S.D., “Quantum symmetry,” in *Lect. Notes in Math.* **1510**, 9–46, Springer 1990.

[GGS93] Gerstenhaber M., Giaquinto A. and Schack S.D., “Construction of quantum groups from Belavin-Drinfel′d infinitesimals,” *Quantum deformations of algebras and their representations*, Israel Math. Conf. Proc. **7**, 45 – 64 Bar-Ilan Univ., Ramat Gan 1993.

[GGSp] Gerstenhaber M., Giaquinto A. and Schaps M., “The Donald-Flanigan Problem for Finite Reflection Groups, *Lett. Math. Phys.* **56** (2001), 41–72.

[GS88] Gerstenhaber M. and Schack S.D., “Algebraic cohomology and deformation theory,” in *Deformation Theory of Algebras and Structures and Applications* (M. Hazewinkel and M. Gerstenhaber Eds.), NATO ASI Ser. C **247**, 11–264, Kluwer Acad. Publ., Dordrecht 1988.

[GS90] Gerstenhaber M. and Schack S.D., “Bialgebra cohomology, deformations, and quantum groups,” *Proc. Natl. Acad. Sci. USA*, **87** (1990) 478–481.

[GS90b] Gerstenhaber M. and Schack S.D., “Algebras, bialgebras, quantum groups, and algebraic deformations,” in *Deformation theory and quantum groups with applications to mathematical physics* (J. Stasheff and M. Gerstenhaber, Eds.), Contemporary Math **134**, 51–92, Amer. Math. Soc., Providence, RI 1992.

[G92] Giaquinto A., “Quantization of tensor representations and deformation of matrix bialgebras,” *J. Pure Appl. Algebra* **79** (1992), 169–190.

[GH98] Giaquinto A. and Hodges T., “Nonstandard solutions of the Yang-Baxter equation,” *Lett. Math. Phys.* **44** (1998), 67–75.

[GZ95] Giaquinto A. and Zhang J.J., “Quantum Weyl Algebras,” *Jour. of Algebra* **176** (1995), 861–881.

[GZ98] Giaquinto A. and Zhang J.J., “Bialgebra actions, twists, and universal deformation formulas,” *J. Pure Appl. Algebra* **128** (1998), 133–151.

[Gr46] Groenewold H., “On the principles of elementary quantum mechanics,” *Physica* **12** (1946), 405–460.

[Gr55] Grothendieck A., “Produits tensoriels topologiques et espaces nucléaires,” *Mem. Amer. Math. Soc.*, No 16 (140p.), 1955.

[Gu83] Gutt S., “An explicit *-product on the cotangent bundle of a Lie group, *Lett. Math. Phys.* **7** (1983), 249–258.

[Hi03] Hinich V., “Tamarkin’s proof of Kontsevich formality theorem,” *Forum Mathematicum* **15** (2003), 591– 61.

[HKR] Hochschild G., Kostant B. and Rosenberg A., “Differential forms on regular affine algebras,” *Trans. Am. Math. Soc.* **102** (1962), 383–406.

[WI53] Inönü E. and Wigner E.P., “On the contraction of groups and their representations,” *Proc. Nat. Acad. Sci. U. S. A.* **39** (1953), 510–524.

[Ji85] Jimbo M., “A q-difference algebra of $\mathscr{U}(\mathfrak{g})$ and the Yang-Baxter equation,” *Lett. Math. Phys.* **10** (1985), 63–69.

[Kb98] Karabegov A.V., “Cohomological classification of deformation quantizations with separation of variables,” *Lett. Math. Phys.* **43** (1998), 347–357; “Berezin’s quantization on flag manifolds and spherical modules,” *Trans. Amer. Math. Soc.* **350** (1998), 1467–1479.

[Ks94] Karasev M. “Formulas for non-commutative products of functions in terms of membranes and strings I,” *Russian J. Math. Phys.* **2** (1994), 445–462.

[KS58] Kodaira K. and Spencer D.C., “On the variation of almost-complex structure,” *Algebraic geometry and topology. A symposium in honor of S. Lefschetz*, 139–150. Princeton University Press, Princeton, N. J. (1957). “On deformations of complex analytic structures I, II,” *Ann. of Math.* **67** (1958), 328–466; “III Stability theorems for complex structures,” *ibid.* **71** (1960), 43–76.

[Ko97] Kontsevich M., “Deformation Quantization of Poisson Manifolds,” *Lett. Math. Phys.* **66** (2003), 157–216 (`q-alg/9709040`).

[KoR] Kontsevich M. and Rosenberg, A. L., “Noncommutative smooth spaces,” *The Gelfand Mathematical Seminars, 1996–1999*, 85–108, Gelfand Math. Sem., Birkhäuser Boston 2000 (`math.AG/9812158`).

[Ko99] Kontsevich M., “Operads and Motives in Deformation Quantization,” *Lett.Math.Phys.* **48** (1999), 35–72 (`math.QA/9904055`).

[KoS] Kontsevich M. and Soibelman Y., “Deformations of algebras over operads and the Deligne conjecture,” in: G. Dito and D. Sternheimer (eds.), *Conférence Moshé Flato 1999*, Math. Phys. Stud. **21**, 255–307, Kluwer Acad. Publ., Dordrecht 2000 (`math.QA/0001151`).

[Ko01] Kontsevich M., “Deformation quantization of algebraic varieties,” *Lett. Math. Phys.* **56** (2001), 271–294 (`math.AG/0106006`).

[KZ01] Kontsevich M. and Zagier D., “Periods,” in *Mathematics unlimited—2001 and beyond*, 771–808, Springer, Berlin 2001.

[KLM99] Kulish P.P., Lyakhovsky V. D. and Mudrov A. I., “Extended Jordanian twists for Lie algebras,” *J. Math. Phys.* **40** (1999), 4569–4586.

[KR81] Kulish P.P. and Reshetikhin N.Yu., “Quantum linear problem for the sine-Gordon equation and higher representations,” *Zap. Nauch. Sem. LOMI* **101** (1981), 101–110 (English translation in *Jour. Sov. Math.* **23** (1983), 24–35).

[LP93] M. Lesimple and G. Pinczon, “Deformations of representations of Lie groups and Lie algebras,” *J. Math. Phys.* **34** (1993), 4251–4272.

[Ln67] Lévy-Nahas M., “Deformations and contractions of Lie algebras,” *J. Math. Phys.* **8** (1967), 1211–1222.

[Li75] Lichnerowicz A., “Variété symplectique et dynamique associée à une sous-variété,” *C. R. Acad. Sci. Paris* Sér. A-B **280** (1975), A523–A527.

[LM88] Lichnerowicz A. and Medina A., “On Lie groups with left-invariant symplectic or Kählerian structures,” *Lett. Math. Phys.* **16** (1988), no. 3, 225–235.

[Lo69] Loos O., *Symmetric Spaces*, Benjamin, New-York, 1969.

[LR91] Lu J-H. and Ratiu T., "On the nonlinear convexity result of Kostant," *Jour. Amer. Math. Soc.* **4** (1991), 349–363.

[LW90] Lu J-H. and Weinstein A., "Poisson Lie groups, dressing transformations, and Bruhat decompositions," *J. Diff. Geom.* **31** (1990), 501–526.

[LS02] Lyakhovsky, V. D. and Samsonov, M. E., "Elementary parabolic twist," *J. Algebra Appl.* **1** (2002), 413–424.

[ML75] Mac Lane S., *Homology*. Springer-Verlag, 1975.

[Ma86] Maillard J.M., "On the twisted convolution product and the Weyl transform of tempered distributions," *J. Geom. Phys.* **3** (1986), 231–261.

[Ma04] Maillard J.M., "Star exponentials for any ordering of the elements of the inhomogeneous symplectic Lie algebra," *J. Math. Phys.* **45** (2004), 785–793.

[Mj90] Majid S., "Physics for algebraists: Non-commutative and non-cocommutative Hopf algebras by a bicrossproduct construction," *J. Algebra* **130** (1990), 17–64.

[Mj92] Majid S., "Tannaka-Krein theorem for quasi-Hopf algebras and other results," in *Deformation theory and quantum groups with applications to mathematical physics* (Amherst, MA, 1990), *Contemp. Math.*, **134** (1992), 219–232.

[Mj93] Majid S., "Braided matrix structure of the Sklyanin algebra and of the quantum Lorentz group," *Comm. Math. Phys.* **156** (1993), 607–638. See also:

[Mj02] Majid S., *A quantum groups primer*, London Mathematical Society Lecture Note Series **292**, x+169 pp., Cambridge University Press, Cambridge, 2002.

[MZ96] Martin C. and Zouagui M., "A noncommutative Hopf structure on $\mathscr{C}^\infty[SL(2,\mathbf{C})]$ as a quantum Lorentz group," *J. Math. Phys.*, **37** (1996), 3611–3629.

[MM65] Milnor J. and Moore J., "On the structure of Hopf algebras," *Ann. Math.* **81** (1965), 211–264.

[Mn77] Molnar R., "Semi-direct products of Hopf algebras," *J. Algebra* **47** (1977), 29–51.

[Mr86] Moreno C., "Invariant star products and representations of compact semi-simple Lie groups," *Lett. Math. Phys.* **12** (1986), 217–229.

[Mo49] Moyal J.E., "Quantum mechanics as a statistical theory," *Proc. Cambridge Phil. Soc.* **45** (1949), 99–124.

[Na98] Nadaud F., "Generalized deformations, Koszul resolutions, Moyal Products," *Reviews Math. Phys.* **10** (1998), 685–704, "Generalized Deformations and Hochschild Cohomology," *Lett. Math. Phys.* **58** (2001), 41–55.

[Nb73] Nambu Y., "Generalized Hamilton dynamics," *Phys. Rev.* **D7** (1973), 2405–2412.

[OSWZ] Ogievetsky O., Schmidke W., Wess J., and Zumino B., "Six generator q-deformed Lorentz algebra," *Lett. Math. Phys.* **23** (1991), 233–240.

[Pe79] Pedersen G., *C*-algebras and their automorphism groups*, London Mathematical Society Monographs **14**, Academic Press, London–New York 1979.

[Pi97] Pinczon G., "Non commutative deformation theory," *Lett. Math. Phys.* **41** (1997), 101–117.

[Pf99] Pflaum M., "Deformation quantization on cotangent bundles," *Rep. Math. Phys.* **43** (1999), 291–297.

[PW90] Podleś P. and Woronowicz S.L., "Quantum deformation of Lorentz group," *Comm. Math. Phys.* **130** (1990), 381–431.

[Re90] Reshetikhin N.Y., "Multiparameter quantum groups and twisted quasitriangular Hopf algebras," *Lett. Math. Phys.* **20** (1990), 331–335.

[RSts] Reshetikhin N.Y. and Semenov-Tian-Shansky M., "Quantum R-matrices and factorization problems," *J. Geom. Phys.* **5** (1989), 533–550.

[Ri89] Rieffel M., "Deformation Quantization of Heisenberg Manifolds," *Commum. Math. Phys.* **122** (1989), 531–562.

[Ri93] Rieffel M., "Deformation quantization for actions of $\mathbf{R}^d$," *Mem. Amer. Math. Soc.* **106** (1993), no. 506.

[Sa61] Saletan E.J., "Contraction of Lie groups," *J. Math. Phys.* **2**, (1961), 1–21 and 742.

[Sch99] Schedler, T., "Verification of the GGS conjecture for $n \leq 12$," `math.QA/9901079`.

[Sch00] Schedler T., "Proof of the GGS conjecture," *Math. Res. Lett.* **7** (2000), 801–826.

[Sc97] Schmid W., "Character formulas and localization of integrals," in *Deformation theory and symplectic geometry, Proceedings of Ascona meeting, June 1996* (D. Sternheimer, J. Rawnsley and S. Gutt, Eds.), Math. Physics Studies **20** 259–270, Kluwer Acad. Publ., Dordrecht 1997.

[SWZ] Schmidke W., Wess J. and Zumino B., "A q-deformed Lorentz algebra," *Z. Phys. C* **52** (1991), 471–476.

[Se51] Segal I.E., "A class of operator algebras which are determined by groups," *Duke Math. J.* **18** (1951), 221-265.

[Sts] Semenov-Tian-Shansky M., "Poisson Lie groups, quantum duality principle, and the quantum double," in *Mathematical aspects of conformal and topological field theories and quantum groups* (South Hadley, MA, 1992), *Contemp. Math.* **175**, 219–248. Amer. Math. Soc., Providence, RI 1994.

[St91] Stolin A., "On rational solutions of Yang-Baxter equation for $\mathfrak{sl}(n)$," *Math. Scand.* **69** (1991), 57–80.

[Sw68] Sweedler M.E., "Cohomology of algebras over Hopf algebras," *Trans. Amer. Math. Soc.* **133** (1968), 205–239.

[Sw69] Sweedler M.E., *Hopf algebras*, W.A. Benjamin, 1969.

[Ta98] Tamarkin D., "Another proof of M. Kontsevich formality theorem," `math.QA/9803025`.

[Ta03] Tamarkin D., "A formalism for the renormalization procedure," `math.QA/0312219`.

[Tm39] Teichmüller O., "Extremale quasikonforme Abbildungen und quadratische Differentiale," *Abh. Preuss. Akad. der Wiss. Math.-naturw. Klasse* **22** (1939), 1–97.

[Tr67] Trèves F., *Topological vector spaces, distributions and kernels*, xvi+624 pp., Academic Press, New York–London 1967.

[UU84] Unterberger A. & J. "Quantification et analyse pseudo-différentielle," and "La série discrète de $\mathrm{SL}(2,\mathbb{R})$ et les opérateurs pseudo-différentiels sur une demi-droite," *Ann. Sci. Éc. Norm. Sup.* (4) **21** (1988), 133–158 and **17** (1984), 83–116.

[UU94] Unterberger A. and Upmeier H., *Pseudodifferential analysis on symmetric cones*, Studies in Advanced Mathematics, CRC Press, Boca Raton FL 1996; "The Berezin transform and invariant differential operators," *Comm. Math. Phys.* **164** (1994), 563-597.

[We94] Weinstein A., "Traces and triangles in symmetric symplectic spaces," *Symplectic geometry and quantization*(Sanda and Yokohama, 1993), Contemp. Math. **179** (1994), 261–270.

[We31] Weyl H., *The theory of groups and quantum mechanics*, Dover, New-York 1931. *Gruppentheorie und Quantenmechanik*, Reprint of the second edition [Hirzel, Leipzig 1931] of the original 1927 text, xi+366 pp. Wissenschaftliche Buchgesellschaft, Darmstadt 1977.

[Wi32] Wigner E.P., "Quantum corrections for thermodynamic equilibrium," *Phys. Rev.* **40** (1932), 749–759.