Diagrams of Lie algebras

Murray Gerstenhaber* Anthony Giaquinto† S.D. Schack‡

December 16, 2003

Diagrams of Lie algebras have natural cohomology and deformation theories. The relationship between Lie and Hochschild cohomology makes it possible to reduce these to the associative case, where the Cohomology Comparison Theorem applies. The latter asserts that for every diagram of associative algebras there is a single associative algebra whose cohomology and deformation theory are the same as that of the entire diagram. In particular, if we have a diagram of Lie algebras and a diagram of Lie modules over it, then there is a single associative algebra whose cohomology with coefficients in a certain bimodule is canonically isomorphic to that of the diagram of Lie algebras with coefficients in its diagram of Lie modules.

1 Diagrams of algebras

A diagram of algebras over small category $\mathcal{C}$ is a contravariant functor $\mathbb{A}$ from $\mathcal{C}$ to some category $\mathcal{A}$ of algebras (e.g. associative, Lie, or in general, over some operad). In the simplest (non-trivial) case, an algebra morphism $B \to A$ may be viewed as a diagram over the poset (partially ordered set) $\{0 \to 1\}$, with $B = \mathbb{A}(1), A = \mathbb{A}(0)$. A very special but important case is the inclusion of a subalgebra into an algebra. Here is a geometric example. Let $\mathcal{C}$ be an open covering of a complex analytic manifold $\mathcal{V}$ containing with every pair of open sets their intersection. View it as a category whose objects are the sets themselves with morphisms the inclusions. That is, whenever i and j are objects of $\mathcal{C}$ such that $i \subset j$ then there is a unique morphism $i \to j$. (This category is again a poset.) Letting $\mathbb{A}(i)$ be the ring of holomorphic functions in i and $\mathbb{A}(j) \to \mathbb{A}(i)$ be the restriction morphism gives a diagram of commutative associative algebras the formal aspects of whose deformation theory includes both that of Froelicher–Nijenhuis–Kodaira–Spencer as well as deformations to

*University of Pennsylvania, Department of Mathematics, Philadelphia, PA 19104-6395; mgersten@math.upenn.edu

†Department of Mathematics, Loyola University of Chicago, Chicago, IL 60626-5385; tonyg@math.luc.edu

‡Department of Mathematics, SUNY at Buffalo, Buffalo, NY 14260-2900; schack@acsu.buffalo.edu

non-commutative manifolds. (For complex manifolds it is convenient to take the covering by Stein opens because these have only trivial cohomology. For algebraic varieties one would similarly take the covering by affine opens.) A morphism between smooth or complex varieties gives rise to a morphism between their Lie algebras of vector fields. Among the important examples of diagrams consider an algebra (Lie or associative) with a group of automorphisms. This may be viewed as a diagram over the group considered as a category with but a single object. Every diagram has its cohomology; in the latter case this is equivariant cohomology.

A significant difference between the associative case and others is that in the associative case one has the collection of theorems called the "Cohomology Comparison Theorem" (CCT), cf. [12]. For simplicity assume for the moment that $\mathcal{C}$ is a poset. The CCT then asserts that for every diagram $\mathbb{A}$ of associative unital algebras there is a single algebra $\mathbb{A}!$ and for every diagram $\mathbb{M}$ of $\mathbb{A}$ bimodules there is a single $\mathbb{A}!$ bimodule $\mathbb{M}!$ such that $H^*(\mathbb{A},\mathbb{M})$ is canonically isomorphic to $H^*(\mathbb{A}!,\mathbb{M}!)$. In particular, $H^*(\mathbb{A},\mathbb{A})$ is canonically isomorphic to $H^*(\mathbb{A}!,\mathbb{A}!)$, so the former carries a Gerstenhaber algebra (G-algebra) structure since the latter does. Moreover the deformation theory of the entire diagram $\mathbb{A}$ is identical with that of the single algebra $\mathbb{A}!$. In the Lie case the cohomology of a single algebra with coefficients in itself carries no cup product (the natural definition of the cup product of two cocycles always giving a coboundary), but it does have a graded Lie product which, as in the associative case, 'controls' the deformation theory. One would therefore expect a graded Lie product in the cohomology of a diagram of Lie algebras. We have a candidate but have not proven that it is correct. For a diagram of Lie algebras, however, one would also expect to have, in general, a non-trivial cup product in the cohomology (and hence a full G-algebra structure) whenever the nerve of the underlying category has a non-trivial cup product in its simplicial cohomology. (The next section has an example.)

In the Lie case, we show that if $\mathbb{L}$ is a diagram of Lie algebras and $\mathbb{M}$ a diagram of modules over it, then there is a single *associative* algebra $\mathbb{A}!$ and a single $\mathbb{A}!$ bimodule $\mathbb{M}!$ over it such that $H^*_{\mathrm{Lie}}(\mathbb{L},\mathbb{M})$ is canonically isomorphic to $H^*_{\mathrm{Hochschild}}(\mathbb{A}!,\mathbb{M}!)$. However, when $\mathbb{M}$ is $\mathbb{L}$ itself then $\mathbb{L}!$ is not the same as $\mathbb{A}!$. As a result, while we do exhibit a conjectured graded Lie algebra structure on $H^*(\mathbb{L},\mathbb{L})$ we do not have a candidate for the full G-algebra structure.

To apply the CCT to the Lie case we prove the following basic result: Suppose given a Lie algebra $\mathfrak{g}$ over a coefficient ring k and a $\mathfrak{g}$ module M which we may view also as a left module over the universal enveloping algebra of $\mathfrak{g}$. Let $U = U\mathfrak{g}$ be the universal enveloping algebra and $\epsilon : U \to k$ be its counit. (The kernel I of ϵ is the 'augmentation ideal'.) Make M into a U bimodule M_ϵ by setting $mu = m\epsilon(u) = \epsilon(u)m$ for $m \in M, u \in U$. Then $H^n_{\mathrm{Lie}}(\mathfrak{g}, M) = H^n_{\mathrm{Hoch}}(U, M_\epsilon)$. Recall that for a group G and G module M we have $H^n_{\mathrm{group}}(G, M) = H^n_{\mathrm{Hoch}}(kG, M_\epsilon)$, where kG is the group algebra of G over the coefficient ring k, M is a k-module on which G operates, and $\epsilon : kG \to k$ is again the counit. The Lie and group theories are thus in perfect analogy. When G is a Lie group it should be possible to deduce the Lie result from that for groups. The

Lie result in turn is an immediate consequence of the following proposition due to Hochschild [19]: Let A be an associative k algebra and N, M be left A modules. View $\mathrm{Hom}_k(N, M)$ as an A bimodule by setting $(afb)(n) = a(f(bn)$ for $a, b \in A$ and $f \in \mathrm{Hom}_k(N, M)$. Then $\mathrm{Ext}^n_A(N, M) = H^n_{\mathrm{Hoch}}(A, \mathrm{Hom}_k(N, M))$, where Ext here is k-relative (no restriction when k is a field). An analogous result holds for Tor. (We give here brief proofs of both.) In a subsequent note we will consider the problem of deforming a Lie algebra as one of giving a special kind of deformation of the Rees ring of its universal enveloping algebra.

Recently Frégier [6] has defined cohomology and given a deformation theory for morphisms of Lie algebras. While this is a special case of diagram cohomology, it is useful to see it computed explicitly in the simplest non-trivial case.

2 The cohomology of a diagram of algebras

The Gerstenhaber-Schack definition of the cohomology of a diagram of algebras, although originally stated only for associative algebras, is actually more general. Suppose that we are given a category $\mathcal{A}$ of algebras over some coefficient ring k, and for each algebra A of the category an abelian category of modules (which in the associative case typically are bimodules) such that for every A-module M there are cochain groups $C^n(A, M)$ together with coboundary operators $\delta_{\mathrm{alg}} : C^n \to C^{n+1}$ defining cohomology groups (actually k modules) $H^n(A, M)$. (In the associative case δ_{alg} will be the Hochschild coboundary and in the Lie case that of Chevalley-Eilenberg.) Suppose that these cochain groups are covariant as functors of M, contravariant as functors of A, and that the coboundary operator commutes with the functorial morphisms. The cohomology groups are then again covariant in M and contravariant in A. If we have a diagram $\mathbb{A}$ of $\mathcal{A}$ algebras over a small category $\mathcal{C}$ then by an $\mathbb{A}$ module $\mathbb{M}$ we will mean a diagram of abelian groups over $\mathcal{C}$ such that for each $i \in \mathrm{ob}\,\mathcal{C}$ the group $\mathbb{M}(i)$ is an $\mathbb{A}(i)$ module and for each morphism $\phi : i \to j$ the map $\mathbb{M}(\phi) : \mathbb{M}(j) \to \mathbb{M}(i)$ is a morphism of $\mathbb{A}(j)$ modules where $\mathbb{M}(i)$ is viewed as an $\mathbb{A}(j)$ module by virtue of the morphism $\mathbb{A}(\phi) : \mathbb{A}(j) \to \mathbb{A}(i)$.

To define the GS cohomology groups $H^n(\mathbb{A}, \mathbb{M})$ we need first the nerve of $\mathcal{C}$. A 0-simplex of this complex is just an object i of $\mathcal{C}$. For $q > 0$ a (non-degenerate) q-simplex of the nerve of $\mathcal{C}$ is any q-tuple of composable morphisms $\sigma = (i_0 \to i_1 \to \cdots \to i_n)$ none of which is an identity morphism. We will call i_0 the *domain* of σ, denoted $d\sigma$ and i_q its *codomain*, $c\sigma$. The 0-th and q-th faces of σ are given, respectively, by $\partial_0\sigma = (i_1 \to \cdots \to i_q), \partial_q\sigma = (i_0 \to \cdots \to i_{q-1})$. For $0 < r < q$ define $\partial_r\sigma$ by composing the maps $i_{r-1} \to i_r$ and $i_r \to i_{r+1}$ so that symbolically we have $\partial_r\sigma = (i_0 \to \cdots \to i_{r-1} \to i_{r+1} \to \cdots \to i_q)$. Setting $\partial\sigma = \sum_{r=0}^{q}(-1)^r\partial_r\sigma$ gives a chain complex whose qth group $C_q(\mathcal{C})$ is the set of all formal finite linear combinations of q-simplices. Note that if $\sigma = (i_0 \to \cdots \to i_q)$ then $\mathbb{M}(d\sigma) = \mathbb{M}(i_0)$ is a module over $\mathcal{A}(i_q) = \mathcal{A}(c\sigma)$ by virtue of the composite morphism $(i_0 \to i_q)$. Let $C^{p,q}$ be the k-module of all functions on $C_q(\mathcal{C})$ which send a q-simplex σ to a cochain $\Gamma \in C^p(\mathbb{A}(c\sigma), \mathbb{M}(d\sigma))$. The image of σ under Γ will be denoted Γ^σ. Setting, as before, $\sigma = (i_0 \to \cdots \to i_q)$, those faces $\partial_r\sigma$ with $0 < r < q$ all have the same domain and codomain as σ, but the first and last do not. Let $\varphi = \mathbb{A}(i_{q-1} \to i_q) : \mathbb{A}(i_q) \to \mathbb{A}(i_{q-1})$. Then $\Gamma^{\partial_q\sigma}\varphi$ defined by sending $a_1, \ldots, a_p \in \mathbb{A}(i_q)$ to $\Gamma^{\partial_q\sigma}(\varphi a_1, \ldots, \varphi a_q)$ is again in $C^p(\mathbb{A}(c\sigma), \mathbb{M}(d\sigma))$. Setting $T = \mathbb{M}(i_0 \to i_1) : \mathbb{M}(i_1) \to \mathbb{M}(i_0)$ we also have $T\Gamma^{\partial_0} \in C^p(\mathbb{A}(c\sigma), \mathbb{M}(d\sigma))$. Writing symbolically

$$\Gamma^{\partial\sigma} = T\Gamma^{\partial_0\sigma} + \sum_{r=1}^{q-1}(-1)^r\Gamma^{\partial_r\sigma} + (-1)^q\Gamma^{\partial_q\sigma}\varphi$$

we now have commuting coboundaries, the algebraic $\delta_{\mathrm{alg}} : C^{p,q} \to C^{p+1,q}$ and the simplicial $\delta_{\mathrm{simp}} : C^{p,q} \to C^{p,q+1}$ defined by $(\delta_{\mathrm{simp}}\Gamma)^\sigma = \Gamma^{\partial\sigma}$. Finally, set $C^n(\mathbb{A}, \mathbb{M}) = \oplus_{p+q=n}C^{p,q}$ and define the total coboundary $\delta : C^n \to C^{n+1}$ by $(\delta\Gamma)^\sigma = \delta_{\mathrm{simp}}(\Gamma)^\sigma + (-1)^{\dim\sigma}\delta_{\mathrm{alg}}(\Gamma^\sigma)$. The GS cohomology groups $H^*(\mathbb{A}, \mathbb{M})$ are the cohomology groups of this double complex.

The foregoing definition clearly works just as well for the Lie case as for the associative case. The definition of a deformation of a diagram $\mathbb{A}$ is always the same: a diagram of $k[[t]]$ algebras and $k[[t]]$ linear morphisms between them which when reduced $\pmod t$ gives the original diagram. As shown in [12, §§21.4-.5] one can not, in general, identify infinitesimal deformations with elements of $H^2(\mathbb{A}, \mathbb{A})$ because of possible 'twisting' of the diagram which results from the fact that the algebras in the diagram generally have inner automorphisms (equally meaningful in the associative and Lie cases). This difficulty disappears when the category $\mathcal{C}$ has a terminator, cf. again [12] (and also, of course, if $\mathbb{A}$ is a diagram of commutative associative algebras). In particular, in the simplest non-trivial case, that where $\mathcal{C}$ is the poset $(0 \to 1)$, everything goes precisely as for a single algebra. In this case we are considering a single algebra morphism $\varphi : B \to A$ and module morphism $T : N \to M$ where N is a B module (bimodule in the associative case), M is an A module, and T is a morphism of B modules when M is considered as a B module by virtue of φ. There are precisely three simplices, the 0-simplices (0) and (1) and a 1-simplex denoted simply (01). An n-cochain therefore has three components, $\Gamma = (\Gamma^0, \Gamma^1, \Gamma^{01})$, where $\Gamma^0 \in C^n(A, M), \Gamma^1 \in C^n(B, N)$ and $\Gamma^{01} \in C^{n-1}(B, M)$, with coboundary $\delta\Gamma = (\delta_{\mathrm{alg}}\Gamma^0, \delta_{\mathrm{alg}}\Gamma^1, T\Gamma^0 - \Gamma^1\varphi - \delta_{\mathrm{alg}}\Gamma^{01})$. For Lie algebras the GS cohomology in this case is precisely that of Frégier [6] (although he does not consider cohomology with coefficients in a module). Clearly the poset $\mathcal{C} = (0 \to 1)$ has a terminator and it is easy to verify directly both in the Lie and in the associative cases that an infinitesimal deformation of a diagram $\mathbb{A}$ of algebras is an element of $H^2(\mathbb{A}, \mathbb{A})$.

This section has been a brief summary. Details and references in the case of associative algebras can be found in [12] which summarizes earlier work. For a simple discussion of the case for a single algebra morphism, including an explicit description of the cup product there, see [11]. For the cohomology of a small category one should also see particularly the paper of Baues and Wirsching [2].

We close with an example to show that while the cohomology of a single Lie algebra has only a trivial cup product there may, in general, be a non-trivial one for a diagram of Lie algebras. Consider, for example, the case where $\mathbb{L}$ is the constant diagram over $\mathcal{C}$ with $\mathbb{L}(i) = k$ (viewed as a one-dimensional Lie algebra) for every i and all morphisms $\mathbb{L}(i \to j)$ the identity. Now $C^m(k,k) = 0$ for $m > 1$ (since the cochains are skew), $C^0(k,k) = k$ and also $C^1(k,k) = k$ (these being k-morphisms of k into k, i.e., just multiplication by an element of k). All coboundaries vanish. It follows that if $\Gamma \in C^n(\mathbb{L},\mathbb{L})$ then the only non-zero components of Γ are the Γ^σ for σ of dimensions n or $n-1$. Writing Σ for the nerve of $\mathcal{C}$, we can therefore identify $C^n(\mathbb{L},\mathbb{L})$ with $C^n(\Sigma,k) \oplus C^{n-1}(\Sigma,k)$. Moreover, $(\delta\Gamma)^\sigma$ now is just $\Gamma^{\partial\sigma}$, so the coboundary operator can be identified with $\delta^n \oplus \delta^{n-1}$. Therefore $H^n(\mathbb{L},\mathbb{L}) \cong H^n(\Sigma,k) \oplus H^{n-1}(\Sigma,k)$. Suppose now $(\eta,\eta') \in H^r \oplus H^{r-1}$ and $(\zeta,\zeta') \in H^s \oplus H^{s-1}$. Defining $(\eta,\eta') \smile (\zeta,\zeta') = (\eta \smile \zeta, \eta \smile \zeta' + (-1)^s\eta' \smile \zeta)$ we have $(\eta,\eta') \smile (\zeta,\zeta') = (-1)^{rs}(\zeta,\zeta') \smile (\eta,\eta')$. So in this case, $H^*(\mathbb{L},\mathbb{L})$ does carry a natural (super)commutative cup product which is non-trivial whenever that in $H^*(\Sigma,k)$ is non-trivial (but it carries no graded Lie structure).

3 Relative Tor and Ext are Hochschild groups

That the relative Ext groups (for a single algebebra) are special cases of Hochschild cohomology was already known (at least when the coefficient ring k is a field) to Hochschild [19]. Similarly, in group cohomology relative Tor and Ext are special cases of Hochschild homology and cohomology, respectively, whenever the relative relative and absolute groups coincide, cf. [3, Prop. III.2.2, p.61]. A short proof for Ext using general homological principles is given in [12, §13.7]. Although we need only the case for Ext, we give here proofs for both Ext and Tor taken from the unpublished [15], together with an intuitive interpretation which may make this basic theorem more understandable.

Let A be a unital k algebra and B be a subalgebra with the same unit. A left A module morphism $f : N \to M$ is B *allowable* or B-*split* if there is a left B module morphism $\alpha : M \to N$ such that $f\alpha f = f$, or equivalently, if $\ker f$ and $\operatorname{im} f$ are direct summands of N, M respectively. (Note that the composite of B split morphisms need not be B split.) A bimodule morphism may be split on one side, or, as in the case of the multiplication map $A \times A \to A$ on each side separately but not as an $A^{\mathrm{e}} = A \otimes A^{\mathrm{op}}$ morphism. (Here, as later $\otimes$ denotes $\otimes_k$.) An exact sequence is called B allowable whenever each morphism is. A left A module P is a B relative projective if given any B allowable epimorphism $f : N \to M$ and an arbitrary $g : P \to M$ there is a morphism $h : P \to N$ with $g = fh$. The concept of a B relative projective (or injective) resolution is then clear. Henceforth when $B = k$ we may omit its mention. (In the following we use "$\bullet$" to denote a complex but "$*$" to denote a sequence of groups unconnected by any morphisms.) The categorical bar resolution (cf. [21, p.270]) of N is given by

$$P_\bullet \to N : \cdots \to A^{\otimes(r+1)} \otimes N \xrightarrow{\partial_r} A^{\otimes r} \otimes N \to \cdots \to A \otimes N \to N \to 0$$

where

$$\partial_r(a_0 \otimes \cdots \otimes a_r \otimes n) =$$

$$\sum_{i=0}^{r-1}(-1)^i(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \ldots a_r \otimes n) + (-1)^r(a_0 \otimes \cdots \otimes a_{r-1} \otimes a_r n),$$

$a_i \in A, m \in M$. This is a k relative projective resolution. (The algebra A operates on the leftmost tensor factor.) When $N = A$ it is a resolution of A by A bimodules, in which case the morphism is right A split. If M' is a right A module, then the relative $\mathrm{Tor}^A_\bullet(M', N)$ is the homology of $M' \otimes_A P_\bullet$, and if M is a left A module the relative $\mathrm{Ext}^*_A(N, M)$ is the homology of $\mathrm{Hom}_A(P_\bullet, N)$. When M is an A bimodule, then taking $N = A$ in the bar resolution and viewing all terms as A bimodules, the Hochschild homology $H_*(A, M)$ of A with coefficients in M is the homology of $M \otimes_{A-A} P_\bullet = M \otimes_{A_e} P_\bullet$, and the cohomology $H^*(A, M)$ is that of $\mathrm{Hom}_{A^e}(P_\bullet, M)$. Observe that $N \otimes_k M'$ and $\mathrm{Hom}_k(N, M)$ are A bimodules, where we have noted k for emphasis. (If $f \in \mathrm{Hom}_k(N, M)$ then afb is defined by $(afb)(n) = a(f(bn))$ for $a, b \in A, n \in N$.)

Theorem 1 *Let N, M be left A modules and M' be a right A module. Then there are natural isomorphisms (i)* $\mathrm{Tor}^A_*(M', N) \cong H_*(A, N \otimes M')$ *and (ii)* $\mathrm{Ext}^*_A(N, M) \cong H^*(A, \mathrm{Hom}\,(N, M))$

Proof. (i) In $M' \otimes_A P_\bullet \to M' \otimes_A N$ note that there are natural isomorphisms $M' \otimes_A N \cong A \otimes_{A^e} (N \otimes_k M')$ and $M' \otimes_A A^{\otimes(r+1)} \otimes_k N \cong A^{\otimes r} \otimes_k A^e \otimes_{A^e} (N \otimes_k M')$. Now we can write the bar resolution of A as

$$\hat{P}_\bullet \to A : \cdots \to A^{\otimes r} \otimes_k A^e \to \cdots \to A^e \to A$$

by transposing the first tensor factor A next to the last. As the modules in $\hat{P}_\bullet \otimes_{A^e} (N \otimes_k M') \to N \otimes_k M'$ and in $M' \otimes_A P_\bullet \to M' \otimes_A N$ are naturally isomorphic, we need only verify that their boundary maps are identical, which they are. In effect, $\mathrm{Tor}^A_*(M', N))$ and $H_*(A, N \otimes M')$ are both the homology of the very same sequence, simply read in two different ways.

(ii) In $\mathrm{Hom}_A(P_\bullet, M) \leftarrow \mathrm{Hom}_A(N, M)$ there are natural isomorphisms

$$\mathrm{Hom}_A(N, M) \cong \mathrm{Hom}_{A^e}(A, \mathrm{Hom}_k(N, M))$$

for both sides are just the center of $\mathrm{Hom}_k(N, M)$, i.e., the elements on which left and right operation by all $a \in A$ coincide, and $\mathrm{Hom}_A(A^{\otimes(r+1)} \otimes_k N, M) \cong \mathrm{Hom}_{A^e}(A^{\otimes r} \otimes A^e, \mathrm{Hom}_k(N, M))$. The rest is similar to (i). □

Isomorphism (ii) has a natural interpretation in relative Yoneda cohomology, where $\mathrm{Ext}^n_A(N, M)$ is the set of equivalence classes of allowable n-extensions

$$\mathcal{E} :\ 0 \to M \to M_n \to \cdots \to M_1 \to N \to 0$$

Denoting $\mathrm{Hom}_k(N, M)$ simply by (N, M), the allowability of this sequence implies the exactness of the bimodule sequence

$$(N, \mathcal{E}) : 0 \to (N, M) \to (N, M_n) \to \cdots \to (N, M_1) \to (N, N) \to 0.$$

Pulling this back on the right by the bimodule morphism $\lambda : A \to (N, N)$ sending $a \in A$ to left multiplication by a defines the morphism $(N, -)\lambda : \mathrm{Ext}^*_A(N, M) \to H^*(A, (N, M))$. For the inverse, note that the categorical bar construction insures that every element of $H^n(A, (N, M))$ can be represented by a right A allowable n-extension of A bimodules

$$\mathcal{E} : 0 \to (N, M) \to M_n \to \cdots \to M_1 \to A \to 0.$$

Allowability implies that $\mathcal{E} \otimes_A N$ remains exact. Pushing this forward on the left by the evaluation map $\mu : (N, M) \otimes_A N \to M$ defined by $f \otimes n \mapsto f(n)$ and identifying $A \otimes_A N$ with N gives a left A module n-extension of N by M. This map, $\mu(- \otimes_A N)$, is the inverse to $(N, -)\lambda$. Note that if N is a relative projective, then the theorem asserts that (N, M) is acyclic as an A bimodule.

The rest of this section contains some remarks on homological algebra not essential to this paper but which may be of independent interest. The *relative left global dimension* of an algebra A is the largest n (if such exists) such that there are left A modules N, M with relative $\mathrm{Ext}^n_A(N, M) \neq 0$, and similarly for the right. The *bidimension* of A is the largest n for which there is a an A bimodule M with $H^n(A, M) \neq 0$; this is just the relative left global dimension of A as a left A^e module. When k is a field, the relative dimensions are just the usual (absolute) ones. We immediately have the following

Corollary 1 *The relative left and right global dimensions of a k algebra A are bounded by its bidimension.*□

It follows, for example, that if A is triangular in the sense of [14] i.e., a semidirect product $S \ltimes J$ of a separable subalgebra S and a 'tensor nilpotent' ideal J, i.e., one such that $J \otimes_A \cdots \otimes_A J$ (ν times) $= 0$, then all the above dimensions are strictly less than ν.

The *poset algebra* of a finite poset $\mathcal{I}$ is the algebra of square matrices with rows and columns indexed by $\mathcal{I}$ and with arbitrary elements in the (i, j) place whenever $i \prec j$ in the partial order (or $i \to j$ if $\mathcal{I}$ is considered as a category). If A is such an algebra then $H^*(A, A)$ is naturally isomorphic to the simplicial cohomology of the nerve of $\mathcal{I}$ with coefficients in k. While not explicit in [16], the methods there show that the analogous result holds for homology. (The poset algebra is the same as the algebra $\mathbb{A}!$ defined below in connection with the CCT when all the algebras $\mathbb{A}(i)$ are identical with k.) Theorem 1 reduces the computation of Ext and Tor for poset algebras to generally much easier problems in simplicial cohomology. Since the homology and cohomology of Tic-Tac-Toe algebras (twisted forms of poset algebras, cf [14]) are the same as those of the corresponding poset algebras, the same is true for them.

Finally, note that in the Yoneda theory there is a cup product $\mathrm{Ext}^n_A(N, M) \times \mathrm{Ext}^n_A(M, L) \to \mathrm{Ext}^n_A(N, L)$ obtained by splicing, while in the Hochschild theory the obvious map of cochains induces a cup product

$$H^n(A, (N, M)) \times H^m(A, (M, L)) \to H^{n+m}(A, (N, L))$$

.

Theorem 2 *The isomorphism* $\mathrm{Ext}^*_A(N, M) \to H^*(A, (N, M))$ *preserves cup products.*□

The proof is a simple application of the universal properties of Ext^* and H^*.

4 The cohomology of a single Lie algebra

In their classic work introducing the cohomology of Lie algebras, Chevalley and Eilenberg [5] showed that if G is a real, compact Lie group and $\mathfrak{g}$ its Lie algebra, then the cohomology of $\mathfrak{g}$ with coefficients in the trivial module $\mathbb{R}$ is identical with the real cohomology of G. Since $\mathfrak{g}$ annihilates the module, the first term on the right in the 'classical' complex below does not occur; its first appearance in print seems to be in [20].

In the classical complex, an n-cochain F of $\mathfrak{g}$ with coefficients in M is a totally skew multilinear map $\mathfrak{g} \times \cdots \times \mathfrak{g} \quad (n \text{ times}) \to M$, or equivalently, a linear map $\bigwedge^n \mathfrak{g} \to M$; for $n = 0$ it is simply an element of M. Denoting the k-module of n-cochains by $C^n(\mathfrak{g}, M)$, the coboundary map $\delta : C^n \to C^{n+1}$ is defined by

$$\begin{aligned}(\delta_{\mathrm{Lie}} F)(a_1, \dots, a_{n+1}) = &\sum_i (-1)^{i-1} a_i \circ F(a_1, \dots, \check{a}_i, \dots, a_{n+1}) \\ &+ \sum_{i<j} (-1)^{i+j} F([a_i, a_j], \dots, \check{a}_i, \dots, \check{a}_j, \dots).\end{aligned} \tag{1}$$

Here the a_i are in $\mathfrak{g}$ and $\circ$ denotes the operation of an $a \in \mathfrak{g}$ on an element of M.

We will need the fact that the right side of (1) is just the skew-symmetrized Hochschild coboundary in the following sense. The symmetric group S_n operates on functions of n variables by setting $(\pi F)(x_1, \dots, x_n) = F(a_{\pi^{-1}1}, \dots, a_{\pi^{-1}n})$. Let $\epsilon_n = \sum (-1)^\pi \pi$ where $(-1)^\pi$ is the signum of π and let $e_n = (1/n!)\epsilon_n$ be the skew-symmetrization idempotent. For an associative algebra A and A bimodule M, let $C^n_K(A, M)$ denote the module of those n-cochains of A with coefficients in M which are skew in all the variables. For $a_1, \dots, a_{n+1} \in A$, and $F \in C^n(A, M)$ the right side is precisely $(\epsilon_n \delta_{\mathrm{Hoch}} F)(a_1, \dots, a_{n+1})$, where in place of $a_i \circ m$ we have $[a_i, m] = a_i m - m a_i$ and $[a_i, a_j] = a_i a_j - a_j a_i$. In particular, denoting by $A_{\mathrm{L}}, M_{\mathrm{L}}$ the Lie algebra and Lie module obtained respectively from A and M by the commutator multiplication, we have

Theorem 3 *There is a natural map* $H^*_{\mathrm{Hoch}}(A, M) \to H^*_{\mathrm{Lie}}(A_{\mathrm{L}}, M_{\mathrm{L}})$. □

A more intrinsic definition of Lie cohomology is given by Cartan and Eilenberg, in their foundational work [4]. Let $U = U\mathfrak{g}$ be the universal enveloping algebra of $\mathfrak{g}$ and $\epsilon : U \to k$ be the counit. A $\mathfrak{g}$ module is then the same as a left module over the associative algebra U and k becomes a left U module by setting $u\lambda = \epsilon(u)\lambda$ for $u \in U, \lambda \in k$. Cartan and Eilenberg then make the definition

$$H^n_{\text{Lie}}(\mathfrak{g}, M) = \text{Ext}^n_U(k, M). \tag{2}$$

On the right one should really understand the relative Ext (which does not yet appear in [4]). The coefficient ring k can be arbitrary but they prove the basic result that when k is a field, or more generally, when $\mathfrak{g}$ is free as a k-module, then the following is a (relative) projective resolution of k as a left U module,

$$P_\bullet \to k : \quad \cdots \to U \otimes \bigwedge^n \mathfrak{g} \xrightarrow{\partial} U \otimes \bigwedge^{n-1} \mathfrak{g} \to \cdots \to U \xrightarrow{\epsilon} k \to 0 \tag{3}$$

where

$$\partial(u \otimes (a_1 \wedge \cdots \wedge a_n)) =$$
$$\sum_i (-1)^{i-1} ua_i \otimes (a_1 \wedge \ldots \check{a}_i \cdots \wedge a_n) + \sum_{i<j} (-1)^{i+j} u \otimes ([a_i, a_j] \wedge \ldots \check{a}_i \ldots \check{a}_j \cdots \wedge a_n).$$

For the proof see e.g. [18, pp. 239–243]. This establishes the equivalence of the new definition with the original. We can now apply Hochschild's theorem of the previous section to rewrite the right side of (2) as $H^n(U, \text{Hom}_k(k, M))$. As a k-module $\text{Hom}_k(k, M)$ is just k itself and the definition of the left U module structure is the original operation of U on M. On the right, however, we have $mu = m\epsilon(u)$. Denote by M_ϵ the U bimodule structure obtained from the left U module M by giving it this right U structure. This gives the following, where $H^*_{\text{Lie}}(\mathfrak{g}, M)$ denotes the cohomology computed from the classical complex (1).

Theorem 4 *Let $\mathfrak{g}$ be a Lie algebra which is free as a module over its coefficient ring k, $U = U\mathfrak{g}$ be its universal enveloping algebra and M be a $\mathfrak{g}$ module. Then there are canonical isomorphisms*

$$H^*_{\text{Lie}}(\mathfrak{g}, M) \cong \text{Ext}^n_U(k, M) \cong H^*_{\text{Hoch}}(U, M_\epsilon).\square$$

However, we shall need more. There is another way to see that $H^*_{\text{Lie}}(\mathfrak{g}, M) \cong H^*_{\text{Hoch}}(U, M_\epsilon)$ which simultaneously exhibits the isomorphism between Lie and Hochschild cohomology at the level of cocycles. We continue to assume that $\mathfrak{g}$ is free as a k-module and that $U = U\mathfrak{g}$ is its universal enveloping algebra. Set $\tilde{P}_n = U \otimes \bigwedge^n \mathfrak{g} \otimes U$ and for $n > 0$ define $d_n : \tilde{P}_n \to \tilde{P}_{n-1}$ by

$$d_n(u \otimes (a_1 \wedge \cdots \wedge a_n) \otimes v) =$$
$$\sum_i (-1)^{i-1} (ua_i \otimes (a_1 \wedge \ldots \check{a}_i \cdots \wedge a_n) \otimes v - u \otimes (a_1 \wedge \ldots \check{a}_i \cdots \wedge a_n) \otimes a_i v)$$
$$+ \sum_{i<j} (-1)^{i+j} (u \otimes ([a_i, a_j] \wedge \ldots \check{a}_i \ldots \check{a}_j \cdots \wedge a_n) \otimes v).$$

For $n = 0$ define $d_0 = \epsilon : U \otimes U \to U$ by $u \otimes v \mapsto uv$. Then the following is a projective (in fact free) resolution of U as a bimodule.

$$\tilde{P}_\bullet \to U :$$
$$\cdots \to U \otimes \bigwedge^n \mathfrak{g} \otimes U \xrightarrow{d_n} U \otimes \bigwedge^{n-1} \mathfrak{g} \otimes U \to \cdots \to U \otimes U \xrightarrow{\epsilon} U \to 0 \quad (4)$$

This is a generalization of the usual Koszul resolution of a polynomial ring (the special case in which $\mathfrak{g}$ is abelian). The proof that this is in fact a resolution follows precisely that for (4) (cf again [18]) and so can be omitted. But now taking $\mathrm{Hom}_U(P_\bullet, M)$ and $\mathrm{Hom}_{U-U}(\tilde{P}_\bullet, M_\epsilon)$ give the same result, namely the complex

$$\mathrm{Hom}_k(P_\bullet, M) :$$
$$\cdots \leftarrow \mathrm{Hom}(\bigwedge^n \mathfrak{g}, M) \xleftarrow{\delta} \mathrm{Hom}(\bigwedge^{n-1} \mathfrak{g}, M) \leftarrow \cdots \leftarrow \mathrm{Hom}(\mathfrak{g}, M) \leftarrow M \quad (5)$$

whose cohomology can therefore be read either as $H^*_{\mathrm{Hoch}}(U, M_\epsilon)$ or $H^*_{\mathrm{Lie}}(\mathfrak{g}, M)$. The cochains in this complex are all skew and the coboundary is the skew-symmetrized Hochschild coboundary. (Note that if $\mathfrak{g}$ is abelian, hence U a polynomial ring, and if M is an arbitrary symmetric U module, i.e., if $am = ma$ for all $m \in M, a \in U$, then all the morphisms in (4) are identically zero, so its modules are the cohomology classes.) It follows, in particular, that the cohomology classes in $H^*_{\mathrm{Hoch}}(U, M_\epsilon)$ can be represented by cocycles which are skew as functions of their variables. Further, these have the property that they are completely determined by their values when all arguments are in $\mathfrak{g}$. For the associated graded algebra of U is a polynomial ring over k where by the theorem of Hochschild-Kostant-Rosenberg [7] cohomology classes are represented by skew cocycles which are derivations as functions of each individual variable and hence determined by their values on the generators of the polynomial ring. That is, the corresponding theorem is true for the associated graded algebra. But a filtered algebra such as U is a deformation of its associated graded algebra [9] and all cocycles of a deformation of an algebra are liftings of those of the original [10]. Hence no skew cocycle of U can vanish when all of its variables lie in $\mathfrak{g}$ unless it is identically zero. (This is also evident from (4).)

Theorem 5 *The isomorphism from $H^*_{\mathrm{Hoch}}(U, M_\epsilon)$ to $H^*_{\mathrm{Lie}}(\mathfrak{g}, M)$ is induced by the map taking a skew cocycle representing the class and restricting it to $\mathfrak{g}$.* $\square$

For the $\mathfrak{g}$ module M we can take $\mathfrak{g}$ itself, which has a k-morphism into U; this is an inclusion when $\mathfrak{g}$ is free over k. With this we can define composition of elements of $C^*(\mathfrak{g}, \mathfrak{g})$ giving, after skew-symmetrization, the graded Lie structure on $H^*_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g})$.

5 The diagram algebra and the Cohomology Correspondence Theorem

To every diagram $\mathbb{A}$ of algebras over a small category $\mathcal{C}$ we can assign a single *diagram algebra* $\mathbb{A}!$ the construction of which is best illustrated when $\mathbb{A}$ is just a single morphism $B \xrightarrow{\phi} A$. For the moment, if $b \in B$ let us denote $\phi(b)$ by b^ϕ. Then $\mathbb{A}! = B + A + A\phi$. As a k-module this is just the direct sum of B and two copies of A, the second denoted by $A\phi$ and its elements written as $a\phi, a \in A$. Multiplication is defined by

$$(b_1 + a_1 + a_1'\phi)(b_2 + a_2 + a_2'\phi) = b_1 b_2 + a_1 a_2 + (a_1 a_2' + a_1' b_2^\phi)\phi.$$

One may view $\mathbb{A}!$ as the set of upper triangular matrices of the form $\begin{pmatrix} a & a'\phi \\ 0 & b \end{pmatrix}$. In the general case, suppose that $\sigma = (i_0 \to i_1 \to \cdots \to i_n)$ is a non-degenerate simplex in the nerve of $\mathcal{C}$. To simplify the notation, let ϕ^σ denote the algebra morphism $\mathbb{A}(c\sigma) \to \mathbb{A}(d\sigma)$ associated to the composite morphism $d\sigma = i_0 \to i_n = c\sigma$. Then

$$\mathbb{A}! = \prod_{i \in \mathcal{C}} \coprod_{d\sigma = i} \mathbb{A}(i)\phi^\sigma.$$

We convene that if σ is a 0-simplex, i.e., just an element i of $\mathcal{C}$, then σ is not degenerate but ϕ^σ is the identity morphism of $\mathbb{A}(i)$. It follows that $\mathbb{A}!$ contains the ordinary direct product $\prod_i \mathbb{A}(i)$ and in particular is unital. Note that ϕ^σ will be an identity morphism whenever σ forms a loop, but if $\mathcal{C}$ is a poset then ϕ^σ is completely determined by its domain $d\sigma$ and codomain $c\sigma$.

The multiplication in $\mathbb{A}!$ is now completely determined by the case for a single morphism, since all we need to know is how to form a product $a\phi^\sigma b\phi^\tau$ when $\sigma = (i_0 \to \cdots \to i_n), \tau = (j_0 \to \cdots \to j_m)$ and $a \in \mathbb{A}(i_0), b \in \mathbb{A}(j_0)$. This product is zero unless $\phi^\sigma b$ is meaningful, i.e., unless $j_m = i_0$, in which case it is $ab^{\phi^\sigma}\phi^\rho$ where ρ is the simplex $(j_0 \to \cdots \to j_m = i_0 \to \cdots \to i_n)$. This definition is meaningful for associative, Lie, and other categories of algebras. If we have an $\mathbb{A}$ module $\mathbb{M}$ then we can similarly form $\mathbb{M}!$, which will be a module over $\mathbb{A}!$. The concepts of a morphism of a diagram of algebras over $\mathcal{C}$ and of a morphism of a diagram of modules over a fixed diagram of algebras are clear, as are the functorial properties of "!". When a functor $\mathfrak{F} : \mathcal{D} \rightsquigarrow \mathcal{C}$ is given we can also consider a morphism from a diagram of algebras $\mathbb{B}$ over $\mathcal{D}$ to a diagram $\mathbb{A}$ over $\mathcal{C}$. It is important also that given a diagram $\mathbb{A}$ over $\mathcal{C}$ and a functor $\mathfrak{F}$ from $\mathcal{D}$ to $\mathcal{C}$ there is an obvious pullback diagram $\mathbb{B}$ over $\mathcal{D}$ and a morphism $\mathbb{B} \to \mathbb{A}$ over the functor $\mathcal{F}$.

Recall now that an n-extension $\mathcal{E} : 0 \to M \to M_n \to \cdots \to M_1 \to N \to 0$ of modules over an associative k-algebra A is allowable if each morphism is k-split. For left modules the Yoneda group of equivalence classes of these n-extensions of N by M is isomorphic to the relative $\mathrm{Ext}^n(N, M)$. A morphism $\mathbb{M} \to \mathbb{M}'$ of diagrams of modules over a diagram of algebras $\mathbb{A}$ will be called allowable if for each object i of the underlying category $\mathcal{C}$ the $\mathbb{A}(i)$ module morphism $\mathbb{M}(i) \to \mathbb{M}'(i)$ is k-split. One does not require that the splittings give a commutative diagram. The Yoneda theory applies to extensions of diagrams; when the diagrams are required to be allowable we will speak of the relative Yoneda theory. When the underlying category is a poset, we have the following basic case of the Cohomology Comparison Theorem, cf [13, 12].

Theorem 6 (Special CCT) *Let $\mathcal{C}$ be a poset and $\mathbb{A}$ be a diagram of algebras over $\mathcal{C}$. (i) If $\mathbb{M}, \mathbb{N}$ are left $\mathbb{A}$ modules then there is natural isomorphism*

$$\mathrm{Ext}^n_{\mathbb{A}}(\mathbb{N}, \mathbb{M}) \xrightarrow{\sim} \mathrm{Ext}^n_{\mathbb{A}!}(\mathbb{N}!, \mathbb{M}!)$$

where the left side denotes the module of equivalence classes of relative Yoneda n-extensions and on the right one has the relative Ext. *(ii) If $\mathbb{M}$ is an $\mathbb{A}$ bimodule, then there is a natural isomorphism*

$$H^n(\mathbb{A}, \mathbb{M}) \xrightarrow{\sim} H^n(\mathbb{A}!, \mathbb{M}!). \square$$

The theorem implies that if $\mathcal{C}$ is a poset then $H^n(\mathbb{A}, \mathbb{A})$ carries the structure of a G algebra since $H^n(\mathbb{A}!, \mathbb{A}!)$ does. The operations are somewhat complicated to describe in the general case but for a diagram reduced to a single algebra morphism they are given explicitly in [11].

To extend the CCT to an arbitrary small category, we need the barycentric subdivision $\mathcal{C}'$ of $\mathcal{C}$ as given in [12]. We will not repeat the definition but give its most important properties.

(1) There is a functor $\mathcal{C}' \to \mathcal{C}$ such that if $\mathbb{A}, \mathbb{N}, \mathbb{M}$ are, repectively, any diagram of algebras over $\mathcal{C}$ and left modules over $\mathbb{A}$, and $\mathbb{A}', \mathbb{N}', \mathbb{N}'$ are their pullbacks over $\mathcal{C}'$ then there is a natural isomorphism $\mathrm{Ext}^*_{\mathbb{A}'}(\mathbb{N}', \mathbb{M}') \cong \mathrm{Ext}^*_{\mathbb{A}}(\mathbb{N}, \mathbb{M})$; similarly, if $\mathbb{M}$ is a diagram of $\mathbb{A}$ bimodules then $H^n(\mathbb{A}', \mathbb{M}') \cong H^n(\mathbb{A}, \mathbb{M})$. Briefly, pullback to $\mathcal{C}'$ preserves cohomology. (2) There are no loops in $\mathcal{C}'$, so we can put a partial order on the objects of $\mathcal{C}'$ by setting $i' \prec j'$ whenever there exists a morphism $i' \to j'$. Such a category is called a *delta*. (3) If $\mathcal{C}$ is a delta, then its barycentric subdivision is a poset.

Denoting by $\mathbb{A}''$ the pullback of $\mathbb{A}$ to the second barycentric subdivision and similarly for $\mathbb{N}'', \mathbb{M}''$, the general CCT for diagrams over an arbitrary small category is identical to the special one except that we must replace $\mathbb{A}!$ by $\mathbb{A}''!$, and similarly for modules. Most important for the deformation theory is that composition of cochains behaves properly so that the deformation theory of $\mathbb{A}''!$ (or just of $\mathbb{A}!$ if $\mathcal{C}$ is already a poset) is identical to that of $\mathbb{A}$.

Suppose now that we have a diagram $\mathbb{L}$ of Lie algebras and a diagram $\mathbb{M}$ of Lie modules over it. We can form $\mathbb{L}!$ and $\mathbb{M}!$ which will be, respectively, a single Lie algebra and and a single Lie module over that algebra, but in general $H^n(\mathbb{L}!, \mathbb{M}!) \ncong H^n(\mathbb{L}, \mathbb{M})$. (Consider, for example the case of a single morphism between two one-dimensional Lie algebras, with module equal to the diagram itself.) However, each $\mathbb{L}(i)$ has its associated universal enveloping algebra $U(i)$ and with these we can form the associative diagram $\mathbb{U}$. From the fact that (4) can be read as giving either Lie or Hochschild cohomology (and using the "generalized simplicial bar" resolution of [12]) we then have

$H^*(\mathbb{L},\mathbb{M}) = H^*(\mathbb{U},\mathbb{M}_\epsilon)$ with the obvious notation. We can now apply the CCT to conclude that $H^*(\mathbb{L},\mathbb{M})$ is naturally isomorphic to $H^*(\mathbb{U}''!,\mathbb{M}''_\epsilon!)$ (or simply to $H^*(\mathbb{U}!,\mathbb{M}_\epsilon!)$ if the underlying category is already a poset).

While this transfers certain questions about the deformation of a diagram of Lie algebras to ones about a single associative algebra it leaves open the question of the existence of a graded Lie structure on $H^*(\mathbb{U},\mathbb{L}_\epsilon)$. Returning to the case of a single Lie algebra $\mathfrak{g}$, there does exist a graded Lie 'hook product' on $H^*(\mathfrak{g},\mathfrak{g})$ described at the cochain level as follows. If $f^m \in C^m(\mathfrak{g},\mathfrak{g}), g^n \in C^n(\mathfrak{g},\mathfrak{g})$ then the hook product $f^m\bar{\wedge}g^n \in C^{m+n-1}(\mathfrak{g},\mathfrak{g})$ is defined by

$$f\bar{\wedge}g(a_1,\dots,a_{m+n-1}) = \sum{}'(-1)^\pi f(g(a_{\pi 1},\dots,a_{\pi n}),a_{\pi(n+1)},\dots,a_{\pi(m+n-1)}),$$

where Σ' is the sum over those permutations π ("pure shuffles") such that $\pi 1 < \pi 2 < \cdots < \pi n$ and $\pi(n+1) < \cdots < \pi(m+n-1)$, and where $(-1)^\pi$ denotes the signum of π. Setting

$$[f^m,g^n]^{\bar{\wedge}} = f\bar{\wedge}g - (-1)^{(m-1)(n-1)}g\bar{\wedge}f$$

defines a graded Lie bracket on $C^m(\mathfrak{g},\mathfrak{g})$ which descends to cohomology. (That is, the bracket of cocycles is again a cocycle and of a cocycle and a coboundary is a coboundary, so it is defined on the cohomology classes.) The problem now is to describe what happens to this bracket under the isomorphism $H^*(\mathfrak{g},\mathfrak{g}) \xrightarrow{\sim} H^*(U\mathfrak{g},\mathfrak{g}_\epsilon)$. As a k-module $\mathfrak{g}_\epsilon$ is identical with $\mathfrak{g}$ which in turn may be viewed as a k submodule of U (assuming, as we do, that $\mathfrak{g}$ is free as a k module) so the composition product as originally defined in [8] of an element of $C^*(U\mathfrak{g},\mathfrak{g}_\epsilon)$ into one of $C^*(U,U)$ is well defined. We can therefore form the hook product and its graded commutator, which, in view of (5), descends to the cohomology level and gives the graded Lie product whose existence is assured by the isomorphism of cohomologies. Some low dimensional cases can, however be addressed directly. In particular for a single Lie algebra $\mathfrak{g}$, recall that its module of infinitesimal deformations is $H^2(\mathfrak{g},\mathfrak{g})$ (where Lie cohomology is understood), which is naturally isomorphic to $H^2(U\mathfrak{g},\mathfrak{g}_\epsilon)$. (Here Hochschild cohomology is understood, and $\mathfrak{g}_\epsilon$ is a bimodule over $U = U\mathfrak{g}$.) If $\eta \in H^2(U\mathfrak{g},\mathfrak{g}_\epsilon)$ (and if division by 2 is possible) then the primary obstruction to η viewed as an infinitesimal deformation is its 'square', i.e., its hook product with itself.

For a diagram of Lie algebras $\mathbb{L}$ with associated diagram $\mathbb{U}$ of universal enveloping algebras, one can consider $(\mathbb{L}''!)_\epsilon$ as a k submodule of $\mathbb{U}''!$. This allows composition of cochains, hence a hook product, but there is no single Lie algebra whose universal enveloping algebra is $\mathbb{U}''!$ so Theorem 5 doesn't apply. We conjecture, however, that it gives a right pre-Lie structure whose graded commutator descends to the cohomology and produces a graded Lie product on $H^*(\mathbb{U}''!,\mathbb{L}''!_\epsilon)$. (Computation might be simplified by the methods of [1].) The infinitesimal deformations can again be identified with the elements of $H^2(\mathbb{U}''!,\mathbb{L}''!_\epsilon)$ (because this is identical with $H^2(\mathbb{L},\mathbb{L})$), but computation of the square is now more difficult and we have not verified that it is the primary obstruction.

Another approach to the cohomology of a diagram of Lie algebras uses the Rees ring of the universal enveloping algebra (which has a natural filtration); it will be treated in a separate note.

6 Appendix: Diagrams of complexes

The functoriality of the Hochschild cochain groups $C^*(A, M)$ – contravariant in the algebra A and covariant in the A bimodule M – essentially forces the correct definition of the GS complex when one passes to diagrams of algebras and modules. Suppose now that we have a diagram of cochain complexes, i.e., a contravariant functor $\mathbb{K}$ from a small category $\mathcal{C}$ to the category of cochain complexes (over the coefficient ring k) and (co)chain mappings. In this case we have no obvious functoriality like that of the Hochschild cochain groups to guide us. Nevertheless we should like to assign to this functor a single complex $\mathbb{K}!$ which in a natural way generalizes the classical mapping cone for the case of the simplest example, a single cochain mapping $f : K' \to K$ cf. [21, pp. 46–47]. So suppose that for every object i of $\mathcal{C}$ we have a cochain complex $\mathbb{K}(i) : \cdots \to \mathbb{K}(i)^n \xrightarrow{\delta} \mathbb{K}(i)^{n+1} \to \dots$ and for every morphism $i \to j$ of $\mathcal{C}$ we have a (co)chain mapping $\mathbb{K}(i \to j) : \mathbb{K}(j) \to \mathbb{K}(i)$ which we will denote by f^{ji}. At this point some choices are possible in the definition of $C^{p,q}$; we define it to be the module of all functions which send a q-simplex $\sigma = (i_0 \to \cdots \to i_q) \in C_q(\mathcal{C})$ to an element of $\mathbb{K}(i_0)^p$ whose coboundary lies in $f^\sigma \mathbb{K}(i_q)^{p+1}$ where f^σ is the composite of the $f^{i_{r+1}i_r}$ coming from σ. The image of σ under $\Gamma \in C^{p,q}$ will be denoted again by Γ^σ. As before, those faces $\partial_r\sigma$ with $0 < r < q$ have the same domain and codomain as σ so $f^{i_1 i_0}\Gamma^{\partial_0\sigma}$ and all $\Gamma^{\partial_r\sigma}$ for $0 < r < q$ all lie in $f^\sigma \mathbb{K}{i_q}^p$ but there is generally no way to map $\Gamma^{\partial_r\sigma}$ into that module *except in the special case where $\mathcal{C}$ is a group (or more generally, a groupoid).* In general, therefore, we define

$$\Gamma^{\check{\partial}\sigma} = f^{i_1 i_0}\Gamma^{\partial_0\sigma} + \sum_{r=1}^{q-1}(-1)^r \Gamma^{\partial_r\sigma},$$

where $\check{\partial}$ indicates that the last term in the expected boundary formula has been omitted. We again have commuting coboundaries, the algebraic $\delta_{\text{alg}} : C^{p,q} \to C^{p+1,q}$ coming from the cochain complexes and the (modified) simplicial $\delta_s : C^{p,q} \to C^{p,q+1}$ defined by $(\delta_s\Gamma)^\sigma = \Gamma^{\check{\partial}\sigma}$. Finally, set $C^n(\mathbb{K}) = \oplus_{p+q=n} C^{p,q}$. The total coboundary $\delta : C^n \to C^{n+1}$ by $(\delta\Gamma)^\sigma = \delta_s(\Gamma)^\sigma + (-1)^{\dim\sigma}\delta_{\text{alg}}(\Gamma^\sigma)$ does have square equal to zero so we may define the cohomology groups $H^*(\mathbb{K})$ to be those of this double complex.

Consider now the special case where $\mathcal{C} = (0 \to 1)$ so we have a single chain mapping $f = f^{10} : \mathbb{K}(1) \to \mathbb{K}(0)$. An n-cochain Γ then is a triple $(\Gamma^0, \Gamma^1, \Gamma^{01})$, where $\Gamma^0 \in \mathbb{K}(0)^n, \Gamma^1 \in \mathbb{K}(1)^n$ and $\Gamma^{01} \in \mathbb{K}(0)^{n-1}$ is an element whose coboundary lies in $f\mathcal{K}(1)^n$. We have $\delta(\Gamma^0, \Gamma^1, \Gamma^{01}) = (\delta\Gamma^0, \delta\Gamma^1, f\Gamma^1 - \delta\Gamma^{01})$. One can see from this formula that the cohomology of the complex just constructed has $H^*(\mathbb{K}(0))$ as a direct summand. For more a more general small category $\mathcal{C}$ this will be the case for any object i of $\mathcal{C}$ for which there is no morphism $j \to i$ other than the identity morphism of i. We therefore lose no information by dropping all corresponding terms from our cochains, obtaining a 'reduced' cochain complex. In our example of a single chain map, the cochains are now reduced to (Γ^1, Γ^{01}) the coboundary of which is $(\delta\Gamma^1, f\Gamma^1 - \delta\Gamma^{01})$. For this to be a cocycle it is clear that $\delta\Gamma^{01}$ must lie in $f\mathbb{K}(1)$ so in computing cohomology it is not necessary to impose that condition on cochains in advance. (It forced us to drop the last term in the simplicial boundary, but as will be seen in the next paragraph we would have to do this anyhow.) Our reduced complex in the special case of a single chain map is now precisely the algebraic mapping cone (except that we have chosen to deal with cohomology instead of homology and have taken our choice of signs from that dictated by the simplicial boundary). It has the property that $\mathbb{K}(0)$ is included (with dimension shifted by 1) into it as the subcomplex of all cochains of the form $(0, \Gamma^{01})$ and the quotient by this subcomplex is just $\mathbb{K}(1)$. It is important (see below) that the last term in the simplicial boundary has been dropped. However, as already mentioned, if $\mathcal{C}$ is a groupoid, i.e., a small category in which every morphism is an isomorphism, then one can include the last term in the simplicial coboundary. In particular, if we have a group operating on a chain complex then this will give a form of equivariant cohomology.

If for an n-cochain Γ and q-simplex $\sigma = (i_0 \to i_1 \to \cdots \to i_q)$ we had simply defined Γ^σ to be an element of $\mathbb{K}^{n-q}(i_0)$ and retained all terms in the simplicial boundary then we would not have recovered the mapping cone for the case of a single cochain map $f : \mathbb{K}(1) \to \mathbb{K}(0)$. An n-cochain then would have been a triple $\Gamma = (\Gamma^1, \Gamma^0, \Gamma^{01})$ with $\delta\Gamma = (\delta\Gamma^1, \delta\Gamma^0, f\Gamma^1 - \Gamma^0 - \delta\Gamma^{01})$. The inclusion of the subcomplex of all $(0, 0, \Gamma^0)$ into this complex would now send every cocycle of $\mathbb{K}(0)$ to a coboundary, for if $\delta\Gamma^0 = 0$ then $\delta(-\Gamma^0, 0, 0) = (0, 0, \delta\Gamma^0)$. This illustrates why the simplicial boundary must be modified.

Finally, note that if a cochain map between complexes has arisen from a continuous map between topological spaces then the cohomology of the algebraic mapping cone is the cohomology of the geometric one. This suggests that more generally if our diagram of complexes has arisen from a diagram of topological spaces then the cohomology of the single complex we have constructed should again be that of a single space constructed from the diagram of spaces, at least in the case where the underlying category is a poset. We leave that question open.

References

[1] Füsun Akman. Chicken or egg? A hierarchy of homotopy algebras. *arXiv:math.QA/0306145 v1*, June 2003.

[2] H.-J. Baues and G. Wirsching. Cohomology of small categories. *J. Pure Appl. Algebra.*

[3] K.S. Brown. *Cohomology of Groups*. Springer-Verlag, New York Heidelberg Berlin, 1982.

[4] H. Cartan and S. Eilenberg. *Homological Algebra*. Princeton University Press, Princeton, New Jersey, 1956.

[5] C. Chevalley and S. Eilenberg. Cohomology theory of Lie groups and Lie algebras. *Trans. Amer. Math. Soc.*, 63:85–124, 1948.

[6] Yaël Frégier. A new cohomology theory associated to deformations of Lie algebra morphisms. *arXiv:math.QA/0304323 v1*, April 2003.

[7] B. Kostant G. Hochschild and A. Rosenberg. Differential forms on regular affine algebras. *Trans. Amer. Math. Soc.*, 102:384–408, 1962.

[8] M. Gerstenhaber. The cohomology structure of an associative ring. *Annals of Math.*, 78:267–288, 1963.

[9] M. Gerstenhaber. On the deformation of rings and algebras II. *Ann. of Math.*, 84:950–954, 1967.

[10] M. Gerstenhaber. On the deformation of rings and algebras III. *Ann. of Math.*, 88:1–34, 1968.

[11] M. Gerstenhaber and S. D. Schack. On the cohomology of an algebra morphism. *Journal of Algebra*.

[12] M. Gerstenhaber and S. D. Schack. *Algebraic cohomology and deformation theory*, pages 11–264. In Hazewinkel and Gerstenhaber [17], 1988.

[13] M. Gerstenhaber and S. D. Schack. The cohomology of presheaves of algebras. I: Presheaves over a partially ordered set. *Trans. Amer Math. Soc.*, 310:135–165, 1988.

[14] M. Gerstenhaber and S. D. Schack. *Triangular algebras*, pages 447–498. In Hazewinkel and Gerstenhaber [17], 1988.

[15] M. Gerstenhaber and S.D. Schack. Relative Tor and Ext are Hochschild groups. 1986.

[16] M. Gerstenhaber and S.D. Schack. Simplicial cohomology is Hochschild cohomology. *J. Pure and Applied Algebra*, 48:229–247, 1987.

[17] Michiel Hazewinkel and Murray Gerstenhaber, editors. *Deformation Theory of Algebras and Structures and Applications, NATO Advanced Study Institute, 'Il Ciocco', Castelvecchio-Pascoli, Tuscany, Italy, June 1-14, 1986.* Kluwer, Dordrecht Boston London, 1988.

[18] P. J. Hilton and U. Stammbach. *A Course in Homological Algebra.* Springer-Verlag, New York Heidelberg Berlin, 1971.

[19] G. Hochschild. Relative homological algebra. *Trans. Amer. Math. Soc.*, 82:246–269, 1956.

[20] G. Hochschild and J.-P. Serre. Cohomology of Lie algebras. *Annals of Math.*, 57:591–603, 1953.

[21] S. Mac Lane. *Homology.* Springer-Verlag, New York Heidelberg Berlin, third edition, 1975.